ISSN 0965-5425, Computational Mathematics and Mathematical Physics, 2017, Vol. 57, No. 7, pp. 1225–1240. © Pleiades Publishing, Ltd., 2017.

A Theoretical Measure Technique for Determining 3D Symmetric Nearly Optimal Shapes with a Given Center of Mass[1]

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Received October 5, 2014; in final form, August 10, 2015

Abstract— In this paper, a new approach is proposed for designing the nearly-optimal three dimensional symmetric shapes with desired physical center of mass. Herein, the main goal is to find such a shape whose image in (r, θ)-plane is a divided region into a fixed and variable part. The nearly optimal shape is characterized in two stages. Firstly, for each given domain, the nearly optimal surface is determined by changing the problem into a measure-theoretical one, replacing this with an equivalent infinite dimensional linear programming problem and approximating schemes; then, a suitable function that offers the optimal value of the objective function for any admissible given domain is defined. In the second stage, by applying a standard optimization method, the global minimizer surface and its related domain will be obtained whose smoothness is considered by applying outlier detection and smooth fitting methods. Finally, numerical examples are presented and the results are compared to show the advantages of the proposed approach.

Keywords: artificial control, center of mass, honey-bee-method, outlier detection, radon measure, symmetric three dimensional shape.

DOI: 10.1134/S0965542517070028

1. INTRODUCTION AND THE PROBLEM STATEMENT

Optimal shape design (OSD) is an application-oriented subject whose applications can be found in many engineering branches. For instance, we can address some of them in mechanical engineering, civil engineering, marine industry and chemical engineering (see [1–4]). We remind that symmetry is an important cue for many applications in the real world, including object alignment, recognition, segmentation, computer graphics and geometric processing [5]. In recent years, symmetry information has been used to detect local features in 2D images [6], to guide reconstruction of 2D curves and range scans with missing data [7], to rotate shapes into a canonical coordinate frame [8]. Moreover, the concept of "center of mass" has a wide range of applications in science and technology; it is an important point on an aircraft which significantly affects its stability. For instance, to ensure that an aircraft is stable, the center of mass must fall within specified limits. If the center of mass is ahead of the forward limit, the aircraft will be less maneuverable [9]. As an example nozzles are a sample of symmetric shapes which have specific center of mass. In this paper, we will investigate a more general design of rotating objects which are independent of their physical system.

It is necessary to remind that until now, two kinds of measures have been used in solving shape optimization problems: Young measure and Radon measure. There is an extensive literature, about Young measures that some of the recent ones can be denoted as [10–13]. Also, many applications are arisen in models of elastic crystals [14], [15]) and optimal design (see [16–18]).

On the other hand, in 1986, Rubio in [19] introduced an embedding process for solving optimal control problems governed by ordinary differential equations, using positive Radon measures. Then, it was employed to obtain the optimal control for systems governed by partial differential equations (like [20], [21]). In sequence, since 1999 until now, with the help of this method, different cases of the optimal shape design problems have been solved (a brief report of these kinds of work was given in [22] and we can also emphasize on [23–27]. Unfortunately, very limited number of articles and books about three-dimensional shape optimization are available; however, there are many industrial objects can not be demonstrated via

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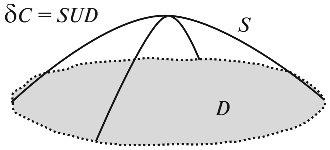

Fig. 1. Unknown 3D shape (unknown surface and its unknown image).

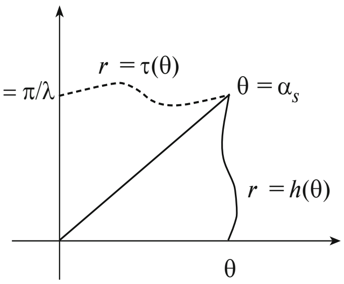

Fig. 2. Unknown domain D.

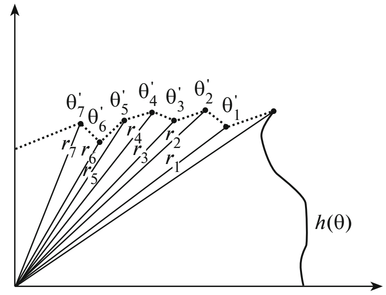

Fig. 3. Unknown boundary of domain D.

two-dimensional tools and a three-dimensional design is needed. Also, 3D optimal shape design methods are problematic because of the following reasons:

(i) The main challenge of most optimization methods is the description of the performed shapes in terms of design variables ([28]).

(ii) Mesh deformation (such as finite element method) was a big difficulty for 3D optimal shape design problems since after a few iterations, the mesh may no longer be feasible. It may cause divergence of the optimization algorithm (see [28]).

(iii) In contrast with 2-D shape optimization problems, parameterizations techniques for 3D problems describe the shape or the shape modifications with a large set of constraints which causes some problems in the convergence of the optimization process (see [29]).

(iv) Iterative methods, such as the level set method, require the objective function to be decreased; but its main drawback was the possibility of falling into a local (and non-global) minima if the initialization was too far from a global minimum [30].

The main goal of this paper is to extend the last above-mentioned method for designing unknown general symmetric three dimensional optimal shape with an unknown image and a given center of mass. We emphasize that this method does not depend on an initial shape or value and can also cover the above mentioned difficulties. Here, the unknown bounded shape C is symmetric in cylindrical coordinate with respect to (r, z)-plane and has a specified center of mass placed on the top of the plane (r, θ). The boundary of the shape includes the unknown surface S with equation $z = f(r, \theta)$ and its unknown image, in (r, θ) plane is the region D (see Fig. 1); that is, a bounded region with a piecewise-smooth, closed and simple boundary ∂D which consists of a fixed and a variable part (see Fig. 2). Due to the nature of the surface S (smooth and continuous), we assume that the function $z = f(r, \theta)$ is absolutely continuous.

The variable part of ∂D is located in $\alpha_1 \le \theta \le \frac{\pi}{2}$ region, where the value α_1 is given. Also, the fixed part, $h(\theta)$, is located in $0 \le \theta \le \alpha_1$ region ($h(\theta)$ is a given continuous function and $\tau(\theta)$ is unknown). We intend to find the nearly optimal unknown surface S and the nearly optimal unknown region D simultaneously, so that a given performance criteria is minimized on C. Furthermore, in general, a curve can be approximated by broken lines so that $\tau(\theta)$ (and hence D) can be approximated with a number M of its points (corners of broken lines belonging to $\tau(\theta)$) which will be called the M-representation of D. For a fixed number M, without losing generality, the points in the M-representation set can have the fixed θ-components like $\theta_i = \theta'_i$, $i = 1,2,\ldots, M$ (see Fig. 3). Hence, each admissible M-representation set called D_M can be characterized by M variables $r_1, r_2, \ldots, r_M$. Therefore, the variable part of ∂D is defined by a finite set of M real variables $(r_1, r_2, \ldots, r_M)$.

From a mathematical point of view, we introduce the set of admissible surfaces as follow:

$$S_A = \{(z = f(r,\theta), D) | \partial C = S \cup D, D \in D_M, (\bar{x}, \bar{y}, \bar{z}) \in C,\ z_{\min} \le z \le z_{\max}\},$$

where the region $D \subset R^2$ and ∂D are defined as follow:

$$D = \begin{cases} 0 \le r \le h(\theta), & 0 \le \theta < \alpha_1, \\ 0 \le r \le \tau(\theta), & \alpha_1 \le \theta < \dfrac{\pi}{2}, \end{cases} \qquad \partial D = \begin{cases} r = h(\theta), & 0 \le \theta < \alpha_1, \\ r = \tau(\theta), & \alpha_1 \le \theta < \dfrac{\pi}{2}, \end{cases}$$

also, $(\overline{x}, \overline{y}, \overline{z})$ is the given center of mass which is defined as follow [31]:

$$\overline{x} = \frac{\int_C x dV}{\int_C dV}, \quad \overline{y} = \frac{\int_C y dV}{\int_C dV}, \quad \overline{z} = \frac{\int_C z dV}{\int_C dV}.$$

The goal is to find an admissible surface which minimizes the given functional $I(S, D)$. So, the problem, that we call (P_1), can be classified as follow:

$$\begin{gathered}(P_1)\colon \min\ I(S,D) = \int_S f_0 d\sigma,\\ S \in S_A, \quad D \in D_M,\\ \text{S. to: } \partial C = S \cup D,\\ (\overline{x}, \overline{y}, \overline{z}) \in C \quad (\text{centerofmass})\\ z_{\min} \le z \le z_{\max},\end{gathered} \tag{1.1}$$

where f_0 is a piecewise continuous function on S. In the first step of the solution, we will obtain a solution of (P_1) for a fixed admissible domain in class D_M. So, we will be able to approximate the value of $I(S, D)$ for any given domain $D \in D_M$ that includes fixed boundaries of D and the unknown boundary which is approximated by the broken lines [32]. Thus, if $M \to \infty$, a sequence $\{D_M^*\}$ of optimal domains tends to a domain D [32] (see Fig. 3).

It is necessary to remind that, in determining domain $D \in D_M$, in addition to the number of boundary points (θ_i, r_i), their suitable distribution on the unknown boundary is also of significance. This fact will be considered by assigning, in a prescribed manner, constant values to $\theta_1, \theta_2, \ldots, \theta_M$ (direction) from a dense subset (see Fig. 3).

Since $d\sigma = \sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1} r dr d\theta_i$ [31], one can state the problem (P_1) mathematically in cylindrical coordinates as:

$$\begin{gathered}(P_2)\colon \min\ I(S,D) = \iint_D f_0(r,\theta,z)\sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1}\, r dr d\theta,\\ f_r, f_\theta, f_{r\theta}, \quad D \in D_M\\ \text{S. to: } \partial C = S \cup D;\\ (\overline{x}, \overline{y}, \overline{z}) \in C;\\ z_{\min} \le z \le z_{\max}.\end{gathered} \tag{1.2}$$

It is difficult to identify a classical solution for the general case of problem (1.2); thus usually it has been tried to find a weak solution of the problem, which is more applicable in our work. The main idea in this replacement, is to change the problem into the variational form and an equivalent optimal control problem to the original problem is obtained. Then, a measure theoretical approach and two stage approximations are used to convert the optimal control problem to a finite dimensional LP. The solution of this LP is used to construct an approximate solution for the original problem.

2. TRANSFERRING THE PROBLEM INTO AN OPTIMAL CONTROL PROBLEM

In order to transfer the optimal shape design in variational form, we need to define some fundamental concepts. First, to simplify the calculations, as in Cartesian coordinates, we have [31]:

$$\int_C (\overline{x} - x) dV = 0, \quad \int_C (\overline{y} - y) dV = 0, \quad \int_C (\overline{z} - z) dV = 0;$$

therefore, in cylindrical coordinates, we have:

$$\int_C (\bar{x} - x)dV = \iint_D \int_{z_{\min}}^{z=f(r,\theta)} (\bar{x} - r\cos\theta)rdzdrd\theta = \iint_D (\bar{x} - r\cos\theta)(z - z_{\min})rdrd\theta = 0.$$

Without loss of generality, we can assume $z_{\min} = 0$; hence,

$$\int_C (\bar{x} - x)dV = \iint_D \int_{z_{\min}}^{z=f(r,\theta)} (\bar{x} - r\cos\theta)rdzdrd\theta = \iint_D (\bar{x} - r\cos\theta)zrdrd\theta = 0.$$

Similarly, we have

$$\int_C (\bar{y} - y)dV = \iint_D (\bar{y} - r\sin\theta)zrdrd\theta = 0,$$

$$\int_C (\bar{z} - z)dV = \iint_D (\bar{z} - z)zrdrd\theta = 0.$$

To solve the optimal shape problem (P_2), we transfer the problem into a control one by defining artificial controls u_i: $D \to R$, $i = 1–3$ as:

$$u_1 = f_\theta, \quad u_2 = f_r, \quad \text{and} \quad u_3 = f_{r\theta}$$

(it will be observed that, control function $u_3 = f_{r\theta}$ is needed in the definition of function Ψ).

Definition 1. We assume that control functions u_1, u_2, and u_3 belong to the bounded sets U_1, U_2, and $U_3 \subset R$ and the path function is $z = f(r, \theta) : D \in D_M \to A \subset R$. We define $\Omega = D \times A \times U_1 \times U_2 \times U_3$.

Clearly, these controls are not independent of each other and the relationship between them should be considered somehow (especially when in numerical schemes). For this reason, let $G(\theta, r, z)$ be a continuous function on $D \times A$, we have:

$$\frac{\partial G}{\partial \theta} = \frac{\partial G}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial G}{\partial z} f_\theta; \quad \frac{\partial G}{\partial r} = \frac{\partial G}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial G}{\partial z} f_r, \quad \forall G \in C(D \times A).$$

Regarding the density property of polynomials in $C(D \times A)$, one may select functions as polynomials in $C(D \times A)$. Without loss of generality, these functions can be selected as the multiplication of different powers of θ, r, z.

We consider a sphere B so that $D \times A \subset B$. We show the space of real-valued and continuously differentiable functions with the first and the second order continuous derivatives bounded on B up to $C'(B)$.

Definition 2. The quaternary $P = (z, u_1, u_2, u_3)$ is called admissible if:

(1) The function $z = f(r, \theta)$ is absolutely continuous.

(2) The control functions u_1, u_2, and u_3 are Lebesgue measurable functions and take their values on the bounded sets U_1, U_2, and U_3. The set of all admissible quaternaries is denoted by W and now problem (P_3) can be rewritten as follows:

$$(P_3): \min\ I(P, D) = \iint_D f_1(r, \theta, z, u_1, u_2, u_3)rdrd\theta,$$

$$P \in W, \quad D \in D_M,$$

$$\text{S. to: } \iint_D (\bar{x} - r\cos\theta)zrdrd\theta = 0;$$

$$\iint_D (\bar{y} - r\sin\theta)zrdrd\theta = 0;$$

$$\iint_D (\bar{z} - z)zrdrd\theta = 0; \tag{2.3}$$

$$\iint_D \left(\frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial z} f_\theta\right) zrdrd\theta = 0, \quad \forall G \in C(D \times A);$$

$$\iint_D \left(\frac{\partial G}{\partial r} - \frac{\partial G}{\partial z} f_r\right) rdrd\theta = 0, \quad \forall G \in C(D \times A);$$

where in the above, $f_1\,(r, \theta, z, u_1, u_2, u_3) = f_0(r, \theta, z)\sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1}$.

Definition 3. In cylindrical coordinates, considering the unit vectors $(\hat{r}, \hat{\theta}, \hat{z})$, the curl of a function $g \in C(R^3)$ is defined in [33] as:

$$\operatorname{curl} g = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ g_r & rg_\theta & g_z \end{vmatrix} = \left(\frac{1}{r}\frac{\partial g_z}{\partial \theta} - \frac{\partial g_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (rg_\theta)}{\partial r} - \frac{\partial g_r}{\partial \theta}\right)\hat{z}.$$

Since in (P_1), the surface equation is $z - f(r, \theta) = 0$, we have:

$$\operatorname{grad} f = \left(-f_r, \frac{-1}{r} f_\theta, 1\right); \quad k = (0, 0, 1).$$

The following conditions put on the functions as well as sets will serve two important purposes. First, they are reasonable conditions which are usually met when considering classical problems. Second, they will allow us to modify these classical problems and these modifications are more advantageous. In this manner, to ensure the admissibility of quaternary $P = (z, u_1, u_2, u_3)$ obtained after solving the problem, we need to define new functions (see [19, 34, 35]).

We consider set of functions $\psi(r, \theta)$ that is infinitely differentiable inside region D(say $\mathcal{D}(D^0)$) and has compact support. Then, we assume $\varphi\,(r, \theta, z) = z\psi(r, \theta)$ (we will introduce function $\psi(r, \theta)$ in the next part). Hence, we define function Ψ so that the absolute continuous condition of path function can be imposed on the problem. Now, we suppose $F = (\varphi_r, \varphi_\theta, \varphi_z)$, then:

$$\nabla \times F = \left(\frac{1}{r}\frac{\partial \varphi_z}{\partial \theta} - \frac{\partial \varphi_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial \varphi_r}{\partial z} - \frac{\partial \varphi_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r\varphi_\theta)}{\partial r} - \frac{\partial \varphi_r}{\partial \theta}\right)\hat{z}.$$

Since the surface equation is $z = f(r, \theta)$, one can conclude that $\nabla f = \left(-f_r, -\frac{1}{r} f_\theta, 1\right)$, then, according to Stoke's theorem in cylindrical coordinate, we have [15]:

$$\oint_{\partial D} \nabla(F) dr = \int_S \nabla \times F . n d\sigma = \iint_D \nabla \times F . \nabla f dA$$

$$= \iint_D \frac{1}{r}(2(r-1) f_r \psi_\theta + f_\theta + z\psi_\theta + (r-1)(f_{r\theta}\psi + f_\theta \psi_r + z\psi_{r\theta})) rdrd\theta = 0,$$

because ψ has a compact support on D, the right hand side of the above integral is equal to zero. Therefore, we define:

$$\Psi(r, \theta, z, u_1, u_2, u_3) = \frac{1}{r}(2(r-1) f_r \psi_\theta + f_\theta + z\psi_\theta + (r-1)(f_{r\theta}\psi + f_\theta \psi_r + z\psi_{r\theta})).$$

Since each differentiable function with finite derivatives satisfies the Lips-chitz condition and is absolutely continuous, [36], function $\psi(r, \theta)$ is absolutely continuous with respect to each of the independent variables r and θ. Also, with the assumption of the admissibility of P, z is absolutely continuous and controls u_1, u_2, u_3 are lebegue measurable; then, function $z\psi(r, \theta)$ is also absolutely continuous (see [36]). So, function $\Psi(\theta, r, z, u_1, u_2, u_3)$ on the Ω is integrable.

Based on similar reasons for the choice $\psi(r, \theta)$, the third class of functions in $C'(B)$ is selected as the functions that only depend on the independent variables θ and r; we show the set of these functions with $C_1(B)$. In this case, we have:

$$\int_S \frac{1}{\sqrt{\frac{1}{r^2}u_1^2 + u_2^2 + 1}} f(r,\theta)d\sigma = \iint_D f(r,\theta)rdrd\theta \equiv a_f; \quad f \in C_1(B),$$

where a_f is the lebesgue integral of function $f(r, \theta)$ on region D.

3. EXPRESSING THE PROBLEM IN THE MEASURES SPACE

In general, the set of all admissible quaternary, W, may be empty or may not contain an optimal quaternary. Even if set W is nonempty and a minimizing quaternary does exist in it, it may be difficult to characterize the optimal solution. Necessary conditions are not always helpful because the information they give may be impossible to interpret. Also, the optimal quaternary may be very difficult or impossible to estimate numerically. Of course, in a given classical problem, the set of admissible pairs is fixed. If we somehow add elements to it, we change the problem and consider a new one. This is precisely our intention; the basis of this metamorphosis is the fact that an admissible quaternary can be considered as something else; that is, a transformation can be established between the admissible quaternary and other mathematical entities. Thus, for any $P = (z, u_1, u_2, u_3) \in W$, we define the following positive linear functional:

$$\Lambda_p : F \in C(\Omega) \to \iint_D F(r,\theta,z,u_1,u_2,u_3)rdrd\theta. \tag{3.4}$$

This transformation is an injection (see Section 6). Now, based on the Riesz representation theorem (see [36]), since positive linear functional can be represented by a positive regular Borel measure (called Radon measure), there exists a positive radon measure, say μ_p, so that:

$$\mu_p(F) \equiv \int_\Omega F(r,\theta,z,u_1,u_2,u_3)d\mu_p = \Lambda_p(F), \quad \forall F \in C(\Omega).$$

Therefore, problem (P_3) with respect to measure μ_p is defined as (P_4):

$$\begin{gathered}
(P_4)\colon \min\ I(P,D) = \mu_p(f_1), \\
P \in W, \\
\text{S. to: } \mu_p((\overline{x} - r\cos\theta)z) = 0; \\
\mu_p((\overline{y} - r\sin\theta)z) = 0; \\
\mu_p((\overline{z} - z)z) = 0; \\
\mu_p\left(\frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial z} f_\theta\right) = 0, \quad \forall G \in C(D \times A); \\
\mu_p\left(\frac{\partial G}{\partial r} - \frac{\partial G}{\partial z} f_r\right) = 0, \quad \forall G \in C(D \times A); \\
\mu_p(\Psi) = 0.
\end{gathered} \tag{3.5}$$

To find the global solution, we enlarge the underlying space and seek for a measure μ in a subset of $M^+(\Omega)$ (the set of all positive Radon measures on Ω) so that it just satisfies in equations (3.5) (not only those that are represented by the Riesz representation theorem [19]). In this manner, an attempt is made to solve the following problem:

$$\begin{gathered}
(P_5)\colon \min\ I(P,D) = \mu(f_1), \\
\mu \in M^+(\Omega), \\
\text{S. to: } \mu((\overline{x} - r\cos\theta)z) = 0; \\
\mu((\overline{y} - r\sin\theta)z) = 0; \\
\mu((\overline{z} - z)z) = 0;
\end{gathered} \tag{3.6}$$

$$\mu\left(\frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial z} f_\theta\right) = 0, \quad \forall G \in C(D \times A);$$
$$\mu\left(\frac{\partial G}{\partial r} - \frac{\partial G}{\partial z} f_r\right) = 0, \quad \forall G \in C(D \times A);$$
$$\mu(\Psi) = 0.$$

We define the set of all positive Radon measures on Ω satisfying (3.6) as Q. Also we assume that $M^+(\Omega)$ be the set of all positive Radon measures on Ω. Now, if we topologize space $M^+(\Omega)$ by the weak*-topology, it can be seen that Q is compact. According to this topology, functional I: $Q \to R$ defined by (3.6) is a linear continuous functional on a compact set Q; thus, it attains its minimum on Q (see Theorem III.1 in [19]), and so the measure theoretical problem which consists of finding the minimum of functional $\mu(f_1)$ over the subset of $M^+(\Omega)$ possesses a minimizing solution, μ^*, in Q. Problem (3.6) is an infinite dimensional LP problem.

It can be concluded from the above discussions that the problem has an optimal solution in Q. But, it is difficult to determine the exact solution because of the infinite number of constraints and the dimension of the solution space. There is still no known and effective method for solving such infinite measure programming problems without making use of approximation. Therefore, we are looking for an approximation close to the optimal solution. In the next step, we will try to apply suitable approximations which are acceptable. In the next section, we are mainly interested in approximations such as those discussed in [19].

4. APPROXIMATION

It is possible to approximate the solution of infinite linear programming (3.6) by the solution of a finite one. First, we consider the minimization of (3.6) not only over set Q, but also over a its subset called $Q(M_1, M_2, M_3, M_4, M_5)$ and defined by only a finite number of constraints to be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces and then by selecting a finite number of constraints. Let sets $\{x_l : l \in \mathbf{N}\}$, $\{f_{sk} : s, k \in \mathbf{N}\}$, $\{G_i \in C(D \times A) : i \in \mathbf{N}\}$ be total sets of functions in the appropriate spaces. We choose a finite number of functions in each of these sets; then, problem (3.6) in an approximation form would be presented as:

$$\begin{gathered}
\operatorname{Inf} I(P) = \mu(f_1), \\
Q(M_1, M_2, \ldots, M_5) \\
\text{S. to: } \mu((\bar{x} - r_j \cos\theta_j) z_j) = 0; \\
\mu((\bar{y} - r_j \sin\theta_j) z_j) = 0; \\
\mu((\bar{z} - z_j) z_j) = 0; \\
\mu\left(\frac{\partial G_i}{\partial \theta_j} - \frac{\partial G_i}{\partial z_j} f_{\theta j}\right) = 0, \quad i = 1, 2, \ldots, M_1; \\
\mu\left(\frac{\partial G_h}{\partial r_j} - \frac{\partial G_h}{\partial z_j} f_{rj}\right) = 0, \quad h = 1, 2, \ldots, M_2; \\
\mu(\Psi_l) = 0, \quad l = 1, 2, \ldots, M_3; \\
\mu(f_{sk}) = a_{f_{sk}}, \quad s = 1, 2, \ldots, M_4, \quad k = 1, 2, \ldots, M_5.
\end{gathered} \tag{4.7}$$

According to Proposition III.1 of [19], if M_1, M_2, M_3, M_4, M_5, and M_6 tend towards infinity, the optimal solution of problem (4.7) will converge to the optimal solution of problem (P_5) and so these two have the same problem. Although the constraints are finite, the solution space is still infinite. To overcome this problem, we introduce a second phase of approximation.

To determine μ^*, we are mainly interested in approximations such as the one discussed in [19]. By regarding the result of Roseribloom's theorem [37], the optimal measure μ^* has the form $\mu^* = \sum_{j=1}^{N} \beta_j \delta(q_j)$, where δ is a unitary atomic measure with the support of the single points q_j which belong to a dense subset of Ω and β_j is a nonnegative real coefficient. So, the problem was changed into a nonlinear One whose unknowns are β_j and the supporting points q_j for $j = 1, 2, \ldots, N$. Moreover, regarding the last set of equations, the number of constraints is still infinite. The idea of converting the problem into a finite linear programming could be approached by putting a discretization on Ω with nodes $q_j = (r_j, \theta_j, z_j, u_{1j}, u_{2j}, u_{3j}) \in \Omega$ in a dense subset of Ω and by selecting a finite number of constraints. Therefore, the solution of (4.7) can be approximated by the following linear programming problem with positive variables β_j.

$$
\begin{aligned}
&\min I(\beta, D) = \sum_{j=1}^{N} \beta_j f_1(q_j);\\
&\text{S. to: } \sum_{j=1}^{N} \beta_j((\overline{x} - r_j \cos\theta_j) z_j) = 0;\\
&\sum_{j=1}^{N} \beta_j((\overline{y} - r_j \sin\theta_j) z_j) = 0;\\
&\sum_{j=1}^{N} \beta_j((\overline{z} - z_j) z_j) = 0;\\
&\sum_{j=1}^{N} \left(\frac{\partial G_i(q_j)}{\partial \theta_j} - \frac{\partial G_i(q_j)}{\partial z_j} f_{\theta j} \right) = 0, \quad i = 1, 2, \ldots, M_1;\\
&\sum_{j=1}^{N} \left(\frac{\partial G_h(q_j)}{\partial r_j} - \frac{\partial G_h(q_j)}{\partial z_j} f_{rj} \right) = 0, \quad h = 1, 2, \ldots, M_2;\\
&\sum_{j=1}^{N} \beta_j(\Psi_l(q_j)) = 0, \quad l = 1, 2, \ldots, M_3;\\
&\sum_{j=1}^{N} \beta_j(f_{sk}(q_j)) = a_{f_{sk}}, \quad s = 1, 2, \ldots, M_4, \quad k = 1, 2, \ldots, M_5.
\end{aligned}
\tag{4.8}
$$

One may say that, at the moment, we are faced with a large scale problem which may have its own difficulties. But, comparing to the original problem, we have to emphasize that we have obtained something more useful. First, we do not know any thing about the existence of the solution of (1) and the manner of obtaining it. But, now, we indicate that the solution exists (Theorem III.1 in [19]) and it could be characterized approximately well by solving a simple finite linear programming problem (4.8). Although by using this method, the dimensions of the problem sometimes could slightly be enlarged for the sake of very precise computations, there are two points worthy of notice which reveal the problem will not suffer from curse dimension or complexity even in high dimensions. First, due to the particular choice of f_{sk} functions and the right-hand-side value of the second class of constraints, a large number of coefficients matrix elements are zero. This reduces the computations and causes the coefficients matrix to be a kind of sparse. Second, the existence of methods such as the interior point in solving linear programming problems for sparse matrices makes the process of solving the problem easier by decreasing the consumed time as well as the complexity of the computations because in such a case, the number of iterations and the consumed time will be reduced ([38]). In addition, it is noteworthy to remind, that several works have been done in this regard till now (e.g. [35, 20, 25]) and even numerical experiences of these papers indicate that under normal conditions, the problem can easily be solved using popular software such as MATLAB, Maple and the modified Simplex method.

5. THE NEARLY OPTIMAL SHAPE

In the present section, we develop a procedure for finding the optimal value of the same functional over the set of all admissible domains D_M, indeed, we intend to solve the mentioned problem in (1.1) by determining the nearly optimal surface and its related nearly optimal image to obtain the minimum value of the performance criterion $I(S, D)$ on W. Each domain $D \in D_M$, as explained, is determined by a set of finite points (θ_m, r_m), $m = 1, 2, \ldots, M$. Thus, for a given $D \in D_M$, by solving (4.8), the nearly optimal value for

$I(\beta^*, D)$ is found as a function of variables $r_1, r_2, \ldots, r_M$. Consequently, one can define the following vector function:

$$J : (r_1, r_2, \ldots, r_M) \in R^M \to I(\beta^*, D) \in \mathbf{R}.$$

The global minimizer of vector function J, say $(r_1, r_2, \ldots, r_M)$, can be identified by using a suitable search technique (like Honey-Bee-Method [39]). Such method normally needs an initial value (initial domain) for starting the process of minimization. Each time the algorithm calculates a value for J, finite linear programming problem (4.8) should be solved; thus, the optimal coefficients β_j^* are characterized. Whenever it reaches the minimum value, the minimizer $(r_1^*, r_2^*, \ldots, r_M^*)$ (optimal domain D^*) and therefore its associated nearly optimal surface have been obtained. So, the nearly optimal domain and the optimal surface are determined at the same time. This is one of the main advantages of this method. In the next section, we summarize the procedure of constructing nearly optimal control functions u_1, u_2, u_3 and path function z derived from a solution of linear programming problem (4.8). We summarize the procedure of constructing nearly optimal control functions u_1, u_2, u_3 and path function z derived from a solution of linear programming problem (4.8): after solving problem (4.8), we identify the indices n such that the components β_n^* of the extreme point are positive and the corresponding value θ_n and r_n associated with them make $\theta = \theta_n$, $r = r_n$, $u_1(r, \theta) = u_{1n}$, $u_2(r, \theta) = u_{2n}$, $u_3(r, \theta) = u_{3n}$, and $z(r, \theta) = z_n$ as mentioned in [19]. Then, we have nearly optimal points (θ_n, r_n, z_n) and by using curve fitting tool box of MATLAB software, we fit the surface on these points in Cartesian coordinates. In this step, there might be some outliers because of employing approximation method which make the shape non-smooth.

Suppose, for a given data set, except a few of its members, the rest belong to special groups (clusters). A few number of members which do not belong to any cluster are called outliers [40]. Also, even if an outlier is a valid data point and not in error, it may deliver unstable results [41–43]. A variety of methods are available to detect outliers. But, in general, all these methods could be classified into two categories. In the first category called labeling method, each data is assigned a label *consistent* or *outlier*. In the second one which is called method of scoring, a number is assigned to any data (called inconsistency factor). The second kind is more flexible because one can choose a threshold value for the incompatibility factors. In this paper, in numerical examples 1 and 2, we used LoOP algorithm (Local Outlier Probability), which is located in the second group of methods, to reject the outliers and achieve a more suitable and flatter shape [43]. In this method, we reject the data whose corresponding outlier factors are more than 0.4.

6. A NOTE ON CONVERGENCE

In this section, we investigate the convergence of the new proposed method according to the following 3 propositions and a theorem.

Proposition 1. *The transformation $P \to \Lambda_p$ of an admissible quaternary in W into the linear mapping Λ_p defined in section* (3) *is an injection.*

Proof. We must show that if $P_1 \neq P_2$, then $\Lambda_{p_1} \neq \Lambda_{p_2}$. Indeed, if $P_1 = (z_1, u_{11}, u_{21}, u_{31})$ and $P_2 = (z_2, u_{12}, u_{22}, u_{32})$ are different admissible quaternary, a continuous positive function F can be constructed on $C(\Omega)$ so that, the right-hand side of Λ_{p_i} corresponding to P_1 and P_2 are not equal. Then, the linear functional are not equal.

Proposition 2. *Let $Q(M_1, M_2, \ldots, M_5)$ be a subset of $M^+(\Omega)$ consisting of all measures which satisfy constraints* (4.7). *As $M_1, M_2, \ldots, M_5$ tend to infinity, then,*

$$\operatorname*{Inf}_{Q(M_1, M_2, \ldots, M_5)} \mu(f_1) \to \operatorname*{Inf}_{Q} \mu(f_1).$$

Proof. Regarding the density properties of the selected subspaces of appropriated spaces $C'(B)$, the proof is similar to the proof of Proposition III.1 in [19].

Proposition 3. *Measure μ^* in set $Q(M_1, M_2, \ldots, M_5)$ at which the function $\mu \to \mu(f_1)$ attains its minimum has the following form*

$$\mu^* = \sum_{j=1}^{N} \beta_j^* \delta(q_j^*),$$

where δ is an atomic measure, $q_j^* \in \Omega$ and $\beta_j^* \geq 0$, $j = 1, 2, \ldots, N$.

Proof. The above-mentioned proof is similar to that of Proposition III.2 in [19].

Theorem 4.1. *Let the search technique give the global minimizer D^* for J and N, $M_1, M_2, \ldots, M_6 \to \infty$; then, the obtained domain and its related nearly optimal surface defined by* (4.8), (D^*, S^*) *are the optimal solution of* (P_1).

Proof. To prove this theorem, we first prove that problem (4.8) is equivalent to Problem (P_1) when N, $M_1, M_2, \ldots, M_5 \to \infty$.

We remind that, problem (P_2) is the same as problem (P_1) which was presented in cylindrical coordinates. Then, extra constraints were added and problem (P_3) was resulted; these constraints are necessary for a better communication between control variables and they also show the admissibility of the quaternaries. On the other hand, according to Proposition 1, problem (3.5) equals problem (2.3). Furthermore, since set W of admissible quaternary can be considered (by means of the injection transformation in Proposition 1) as a subset of Q, the minimization of problem (3.6) is global; that is, the global minimum of problem (3.5) can be approximated well [19]. Also, as mentioned in Section 3, problem (3.6) has at least one solution. Now, according to Propositions 2 and 3, when $M_1, M_2, \ldots, M_5 \to \infty$, problem (4.8) is equivalent to (3.6). Also, with respect to Proposition III.3 and Theorem III.1 in [19], problem (4.8) has a solution and this solution converges to the solution of problem (3.6) when N is sufficiently large. So, problem (1.1) has at least one solution and its solution can be determined with the method presented in section 5.

Now, if $I(\beta^*, D^*)$ is not an optimal solution of (4.8), then, there will be a domain and its related vector β where $D' \in D_M$ and $I(\beta, D') < I(\beta^*, D^*)$. Let β' be the optimal vector for problem (4.8) defined with respect to the given domain D'; then, $I(\beta', D') \le I(\beta^*, D') < I(\beta^*, D^*)$; that is $J(D') < J(D^*)$. Let $(r_1', r_2', \ldots, r_M')$ be the representation of domain $D' \in D_M$; thus, we have $J(r_1', r_2', \ldots, r_M') < J(r_1^*, r_2^*, \ldots, r_M^*)$. This inequality states that $(r_1^*, r_2^*, \ldots, r_M^*)$ is not the global minimizer of J, which is a contradiction. Thus $I(\beta^*, D^*)$ is the optimal value for functional I and (S^*, D^*) is nearly optimal.

6.1. Total Sets

To restrict the number of constraints (3.6), we considered countable sets of functions whose linear combinations are dense in the specific space. In this section, we attempt to introduce some suitable cases for such sets. In this manner, we explain how one can choose total sets for the constraints of (4.8). Infinitely differentiable functions consist of functions such as exponential and trigonometric functions. However, exponential function can never be zero. Therefore, we make use of trigonometric functions whose linear combinations can make Fourier series for each periodic arbitrary function. We choose these functions in the following way and we consider ψ_{ji}''s as:

$$\begin{aligned}\psi_{1i} &= (r - h(\theta))(\sin(i\pi\theta));\\ \psi_{2i} &= (r - h(\theta))(\cos(i\pi\theta));\\ \psi_{3i} &= (r - h(\theta))(\cos(i\pi\theta)\sin(i\pi\theta)).\end{aligned}$$

Obviously, the linear combinations of these functions are uniformly dense in the space $C_1(\Omega)$, infinitely differentiable inside region D and has compact support (see [35]).

To be able to characterize the optimal coefficients of (4.8), an arbitrary domain will be divided into finite parts and then, an attempt will be made to determine the nearly optimal surface in each part. In this manner, a finite number of angles $\theta = \theta_i$, $i = 1, 2, \ldots, l$ from the uniform dense subset of $[0, \alpha_1)$ (given part of domain D) and $\theta' = \theta_j'$, $j = 1, 2, \ldots, M$ of $[\alpha_1, \pi/2)$ (variable part of domain D) would be considered (see Fig. 4). Then, domain D can be divided into $s = l + M$ parts by half-lines $\theta = \theta_i$ and $\theta' = \theta_j'$. Also, the ith part of D ($i = 1, 2, \ldots, l$) can be approximated by the sector $R_i = \dfrac{h(\theta_i) + h(\theta_{i+1})}{2}$, when $\theta_i \le \theta \le \theta_{i+1}$, the jth part ($j = 1, 2, \ldots, M$) can be approximate by the sector $R_i' = \dfrac{r_i + r_{i+1}}{2}$ when $\theta_i' \le \theta \le \theta_{i+1}'$ and $(r_1, r_2, \ldots,$

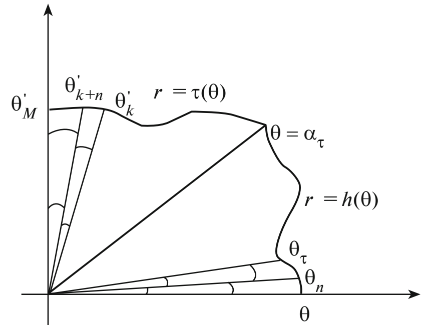

Fig. 4. Sample partition of D for introduce f_{sk} (r, θ).

r_M) is the optimal value in $(\theta'_1, \theta'_2, \ldots, \theta'_M)$, say D_j. Then, if the number of angles is sufficiently large, the union of D_j's can approximate D arbitrarily. So, we consider f_{sk} as follow:

$$f_{sk}(\theta, r) = \begin{cases} 1, & \text{if} \quad \theta \in J_{1s}, \quad r \in J_{2k}, \\ 0, & \text{otherwise}, \end{cases}$$

where J_{1s} and J_{2k} are determined as follow:

$J_{1s} = \left[\dfrac{(s-1)\theta_i}{M_4}, \dfrac{(s)\theta_i}{M_4}\right)$ and $J_{2k} = \left[\dfrac{(k-1)R_i}{M_5}, \dfrac{(k)R_i}{M_5}\right)$ for given part of region D and $J_{1s} = \left[\dfrac{(s-1)\theta'_j}{M_4}, \dfrac{(s)\theta'_j}{M_4}\right)$ and $J_{2k} = \left[\dfrac{(k-1)R'_i}{M_5}, \dfrac{(k)R'_i}{M_5}\right)$ for the unknown part. Hence:

$$\iint_D f_{sk}(\theta, r) r dr d\theta = \int_{r_{k-1}}^{r_k} \int_{\theta_{s-1}}^{\theta_s} r dr d\theta = \frac{1}{2}(\theta_s - \theta_{s-1})(r_k^2 - r_{k-1}^2) \equiv a_{f_{sk}}.$$

7. NUMERICAL EXAMPLES

In this section, by giving some numerical examples, we examine the efficiency of the method explained in the previous sections.

Example 1. A simple computation shows that point (9/5, 0, 0) is the center of mass for the generated rotating shape (around x-axis) by the generator curve $y = \sqrt{x+1}$, $0 \le x \le 3$. To have a comparison, hereby, we intended to find the surface S with image D with unknown boundary $\tau(\theta)$ and the initial and the final points (0,1) and (3,2) so that point (9/5, 0, 0) is its center of mass. We consider the initial curve $\tau(\theta)$-as the line that possesses points (0,1) and (3,2) i.e., $y = 1/3x + 1$. So, in polar coordinate we have $\tau(\theta) = \dfrac{1}{\sin\theta - (1/3)\cos\theta}$. For this reason, we choose: $z_{\min} = 1$, $z_{\max} = 2$, $0 \le \theta \le \pi/2$, $-3/2 \le f_\theta \le 3/2$, $0 \le f_r \le 1$, $-1 \le f_{r\theta} \le 1$.

To discretize $\Omega = D \times A \times U_1 \times U_2 \times U_3$, we chose $M = 16 \times 6 \times 10 \times 8^3$ point in these set by selecting

1. 8 points in U_1 for f_θ: $-3/2$, $-15/14$, $-9/14$, $-3/14$, 3/14, 9/14, 15/14, 3/2;

2. 8 points in U_2 for f_r: 0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1;

3. 8 in U_3 for $f_{r\theta}$: -1, $-5/7$, $-3/7$, $-1/7$, 1/7, 3/7, 5/7, 1;

4. 16 angles in $[0, \pi/2]$ for θ:

$$0, \pi/30, 2\pi/30, 3\pi/30, 4\pi/30, 5\pi/30, 6\pi/30, 7\pi/30, 8\pi/30, 9\pi/30,$$
$$10\pi/30, 11\pi/30, 12\pi/30, 13\pi/30, 14\pi/30, \pi/2;$$

5. 10 value in A [1, 2] for z: 1, 10/9, 11/9, 12/9, 13/9, 14/9, 15/9, 16/9, 17/9, 2.

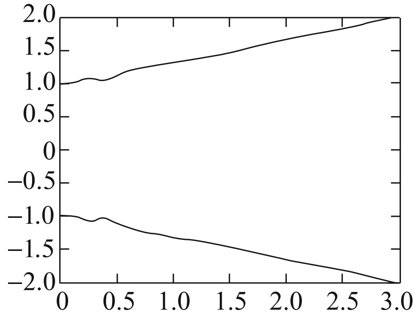

Fig. 5. Nearly optimal domain D in Example 1.

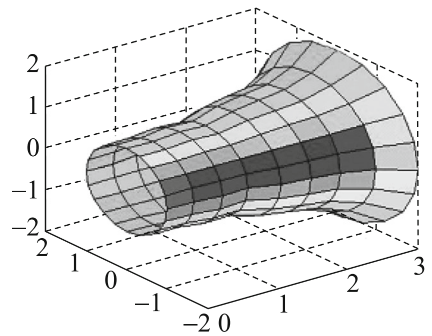

Fig. 6. Rotating surface with generator curve $\tau(\theta)$ in Example 1.

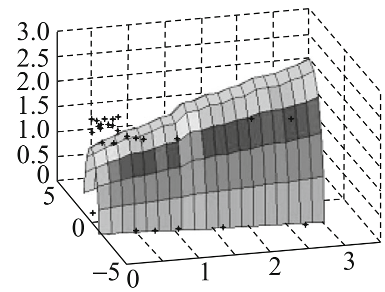

Fig. 7. Nearly optimal surface in Example 1 without rejecting outliers.

To set up the linear programming problem (4.8), for the first set of constraints, we select $M_1 = 2$ and $M_2 = 2$ and choose:

$$G_{1l} = (2\theta_n z_n r_n) - (\theta_n^2 r_n f_{\theta_n}),\ l = 1, 2;$$

$$G_{2k} = (2\theta_n z_n^2 r_n^2) - (2\theta_n^2 r_n^2 z_n f_{\theta_n}),\ k = 1, 2.$$

For $M_3 = 9$, we selected:

$$\begin{aligned}\psi_{1i} &= (r_n - h(\theta_n))(\sin(i\pi\theta_n)),\\ \psi_{2i} &= (r_n - h(\theta_n))(\cos(i\pi\theta_n)),\\ \psi_{3i} &= (r_n - h(\theta_n))(\cos(i\pi\theta_n))(\sin(i\pi\theta_n)), \quad i = 1, 2, 3.\end{aligned}$$

Choose $f_{sk}(\theta, r)$ with $M_4 = 8$ and $M_5 = 3$ (by selecting $1/3R_i'$, $2/3R_i'$, R_i' in variable region of D and $1/3R_i$, $2/3R_i$, R_i in given region for $0 \le \theta \le \pi/2$). So, the linear programming problem by 40 constraints and 491520 variables with a constant objective function was set up.

In 2012, Fakharzadeh et al. [32] dealt with the best standard algorithm to identify the optimal solution for an OSD sample problem governed by an elliptic boundary control problem. Their goal in that paper was to examine and evaluate six different methods according to their ability to find optimality of function. In the mentioned paper, some references (references related to applications or discussions) were given for each method. They conducted a computational examination of several existing derivative free optimization methods to apply solution procedure of OSD problems by shape-measure technique. These methods consist of Random search, Nelder-Mead algorithm, Hook and Jeeves algorithm, Simulated annealing algorithm, Genetic and Honey bee mating optimization algorithm. The results showed that Random search and Honey bee mating.optimization algorithm are more appropriate for use in shape measure method than other algorithms [32]. In this manner, we use Honey Bee Mating optimization algorithm (HBM) to obtain the optimal value of $J(r_1, r_2, \ldots, r_M)$ and the modified Simplex method from MATLAB 7.13 to obtain the optimal value of $I(\beta, D)$.

After solving this problem by 50 iterations, we obtained optimal points (θ_n, r_n, z_n) corresponding to optimal coefficients $\beta_n^* > 0$ in the manner explained in section 4. Then, we fitted a surface on these points with and without rejecting the outliers by using related MATLAB's toolbox. Figures 5 and 6 show the obtained domain and the rotating surface. In addition, due to the symmetry, we can obtain the rotating surface around x-axis by using generating curve $\tau(\theta)$.

Figure 7 shows the obtained nearly optimal surface without rejecting outliers by normal curve fitting procedure from MATLAB and Fig. 8 shows this nearly optimal surface by rejecting outliers with same procedure. As is shown in the figures, in Fig. 7, there are more outlier points which are not laid on the fitted surface.

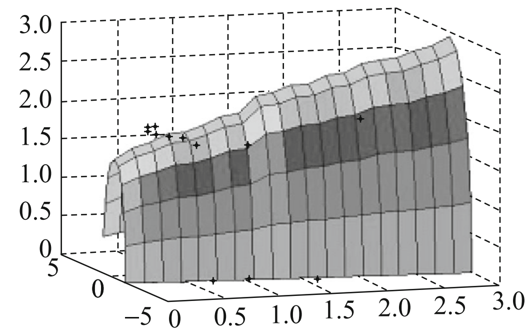

Fig. 8. Nearly optimal surface in Example 1 with rejecting outliers.

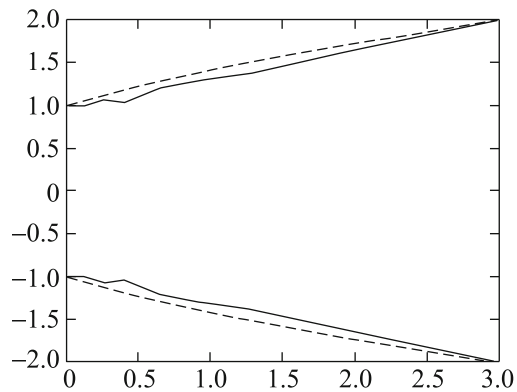

Fig. 9. Nearly optimal domain D and curve $y = \sqrt{x+1}$ in Example 1.

In Fig. 9, the obtained curve with new method is marked in red and the curve $y = \sqrt{x+1}$ is shown in dashed line. The unknown boundary has been approximated by 7, 9, and 12 points and in each case, L^2 error has been calculated using formula ([44])

$$L^2\text{error} = \|y_{\text{opt}} - y_{\text{analytical}}\|_2 = \sqrt{\sum_{i=1}^{n}(y_{i_{\text{opt}}} - y_i(\sqrt{x_i+1}))^2},$$

and L^∞ error has been calculated using formula

$$L^\infty\text{error} = \|y_{\text{opt}} - y_{\text{analytical}}\|_\infty = \max|y_{i_{\text{opt}}} - y_i(\sqrt{x_i+1})|.$$

Tables 1–3 show the values of y_{opt}, $y = \sqrt{x+1}$ at the same x as well as the mentioned errors for these three cases.

Table 1. The values of y_{opt} and $y = \sqrt{x+1}$ at the same x (7 points)

x	y_{opt}	$y = \sqrt{x+1}$	L^2 error	L^∞ error
3.0000	2.0000	2.0000	0.6628	–0.5196
0.7183	0.7913	1.3108		
0.6533	0.9263	1.2825		
0.6279	0.1732	1.2758		
0.3897	0.0090	1.1788		
0.2764	1.1022	1.1297		
0.0000	1.0000	1.0000		

Table 2. The values of y_{opt} and $y = \sqrt{x+1}$ at the same x (9 points)

x	y_{opt}	$y = \sqrt{x+1}$	L^2 error	L^∞ error
3.0000	2.0000	2.0000	0.2455	0.1293
1.8552	1.5973	1.6897		
1.2418	1.3680	1.4973		
0.9118	1.2929	1.3827		
0.6437	1.2028	1.2821		
0.4005	1.0371	1.1834		
0.2676	1.0670	1.1259		
0.1234	0.9996	1.0599		
0.0000	1.0000	1.0000		

Table 3. The values of y_{opt} and $y = \sqrt{x+1}$ at the same x (12 points)

x	y_{opt}	$y = \sqrt{x+1}$	L^2 error	L^∞ error
3.0000	2.0000	2.0000	0.1631	0.0597
2.1854	1.8053	1.7847		
2.1000	1.7857	1.7607		
1.8085	1.7186	1.6758		
1.800	1.7118	1.6733		
1.500	1.6295	1.5811		
1.3018	1.4708	1.5172		
1.2000	1.4473	1.4832		
0.9000	1.3783	1.3784		
0.6502	1.3209	1.2846		
0.6000	1.3093	1.2650		
0.0000	1.0000	1.0000		

As observed, by increasing the number of unknown boundary points, both errors have decreased and hence, the accuracy will increase.

Example 2 (cited from [45, p. 123]). The aim is to find a nearly-optimal surface so that the boundary of D passing the points (0, 1, 0) and (3, 2, 0) and the surface has the minimum area with the center of mass at point (1.9, 0, 0). Thus, the performance criterion is the minimization of $\int_S d\sigma$ with the additional condition $\int_C (\bar{x} - x)dV = 0$ and the other conditions which are similar to those of Example 1.

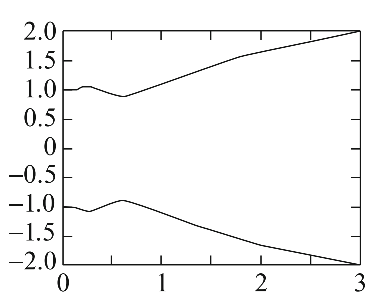

Fig. 10. Nearly optimal domain D in Example 2.

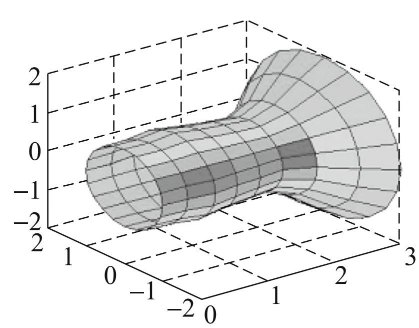

Fig. 11. Rotating surface with generator curve $\tau(\theta)$ in Example 2.

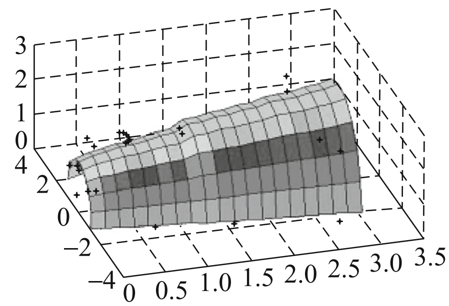

Fig. 12. Nearly optimal in Example 2 without rejecting outliers.

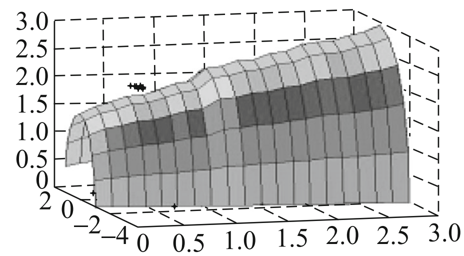

Fig. 13. Nearly optimal surface in Example 2 with rejecting outliers.

In the same way as in Example 1, Figs. 10 and 11 show the obtained domain and the rotating surface by the fitting procedure using MATLAB 7.13 software. Also, Figs. 12 and 13 show the obtained surface by the fitting procedure without and with rejecting the outliers. The value of optimal objective function is 23.6332 and the approximated area of rotating surface around x axis by using generating curve $\tau(\theta)$, equals to 31.7330. Also, comparison the results with the analytical solution from [45] indicates the acceptable of the new method. As is shown in the figures, in Fig. 12, there are more outlier points which are not laid on the fitted surface.

8. CONCLUSION

In this paper, a novel practical approach was proposed to obtain the solution of 3D symmetric shape optimization problems with a given center of mass. First, the problem was transferred into an optimal control frame in a variational representation. Then, in an algorithmic path, the nearly optimal shape was constructed by transferring the problem into a measure space, extending the underlying space, applying two approximation steps and obtaining the optimal surface and its image from the solution of an appropriate finite linear programming problem. Moreover, the optimal value for the general form of the objective function and the nearly optimal shape were determined in an easy way just by applying a standard search technique (by supposing an initial image) and also by implementing the Simplex algorithm perfectly well.

Additionally, in this method, a smoother shape was obtained by rejecting the outlier data and smooth fitting procedures. The method has many advantages including the automatic existence theorem, the linearity of the solution method even for extremely nonlinear problems, the easy imposition of the wished physical properties for the optimal shape and also the generality and simplicity in application for different purposes and systems. Especially, it is practical and accurate enough for systems with nonlinear terms while, accuracy can be improved as much as desired. Furthermore, in the future studies, we will try to solve mechanical problems in 3 dimensions by this new method.

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