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A new method for solving coupled complex matrix equations

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Abstract. In this paper, a new method for solving coupled complex matrix equations is applied. In this method, we change the problem into a real equation system by using the multiplication properties of complex numbers. This new problem can be solved easily. Numerical examples are given to show the efficiency of the new method.

Keywords: Complex matrix, coupled system, imaginary part, Kronecker product, permutation matrix, real part.

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1. Introduction

Matrix linear equations like $AXB = M$, $AXB+CXD = M_1$ and $AXB+CX^TD = M_2$ are very important in linear systems and the methods of solving such equations have been studied in various articles [1, 2, 4]. A method for solving equation systems of

$$\begin{cases} A_1XB_1 = C_1 \\ A_2XB_2 = C_2 \end{cases}$$

is proposed in [11]. Also, two iterative algorithms are proposed for solving the following coupled matrix equations in [3]:

$$\begin{cases} A_1XB_1 + C_1X^TD_1 = M_1 \\ A_2XB_2 + C_2X^TD_2 = M_2 \end{cases}$$

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Ding and Chen [5] introduced a hierarchical gradient iterative as well as a hierarchical stochastic gradient algorithm proving that the parameter estimation errors given by these algorithms converge to zero for all initial values under persistent excitation. In [7], the researchers presented a family of iterative methods for linear systems and studied a least-squares iterative solution to coupled matrix equations by using the hierarchical identication principle and the star product. In [6], a hierarchical identication principle was used to solve the Sylvester and Lyapunov matrix equations. In [8], using the gradient search principle, the researchers studied gradient iterative algorithms for solving Sylvester coupled matrix and general coupled matrix equations. In [10], the scholars obtained Cramers rules for some quaternion matrix equations within the framework of the column and row determinants theory. In [12], the necessary and sufficient condition for the existence of the solution to the matrix equation $AX + X^tC = B$ was examined using the generalized inverse matrix. The above-mentioned studies were briefly reviewed in [3] and we also emphasize [1–3]. In all the mentioned researches, the real coefficient matrix and real unknowns have been considered. In practical cases, such as optimal control and numerical analysis, the coefficient matrix might be complex.

In this article, a new and practical method is proposed for solving the four types of the following complex equations:

$$AXB + CX^*D = M \tag{1}$$

$$\begin{cases} A_1XB_1 + C_1X^*D_1 = M_1 \\ A_2XB_2 + C_2X^*D_2 = M_2 \end{cases} \tag{2}$$

$$AX_1B + CX_2D = M \tag{3}$$

$$\begin{cases} A_1X_1B_1 + C_1X_2D_1 = M_1 \\ A_2X_1B_2 + C_2X_2D_2 = M_2 \end{cases} \tag{4}$$

We denote the set of all $m \times n$ complex matrices by $\mathbf{C}^{m\times n}$. Therefore, the coefficient matrix A_i, B_i, C_i, D_i, the amounts to the right hand side, M_i, and the unknown matrix, X (or X_i), are complex. We use the following notation.

X^* is the conjugate transpose of matrix X, $R(A)$ is a matrix with the same dimension as matrix A, which only includes the real part of elements in matrix A. $I(A)$ is a matrix with the same dimension as matrix A, which only includes the imaginary part of elements in matrix A. For example, if

$$A = \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix},$$

then

$$R(A) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \qquad \text{and} \qquad I(A) = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

Also, for the arbitrary matrix, $Vec(X)$ is a vector with mn elements obtained by stacking the columns of matrix X. For the two given matrices $X \in \mathbf{C}^{m\times n}$ and $Y \in \mathbf{C}^{l\times k}$, Kronecker product $X \otimes Y$ is the $ml \times nk$ matrix that is calculated by $X \otimes Y = [x_{ij}Y]$. Therefore, $Vec(AXB) = (B^T \otimes A)Vec(X)$, where A, B and X are matrices with the proper dimensions [9].

2. Presenting the problem and the solution method

In this part, a method is proposed for solving the complex equation systems. By using the multiplication properties of complex numbers, we will turn the equation system $AXB + CX^*D = M$ into a linear and real equation system like $KX = D$ which is easily solvable with several methods. Then, we will also turn the complex equation systems (2), (3) and (4) into real systems $K_1X_1 + K_2X_2 = D_2$, based on the proposed method.

Theorem 2.1 The coupled complex equation system $AXB+CX^*D = M$ can be turned into a real equation system $KX = D$.

Proof. If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers, then $R(z_1) = a_1$, $R(z_2) = a_2$, $I(z_1) = b_1$ and $I(z_2) = b_2$. According to the multiplication properties of complex numbers, we have

$$R(z_1z_2) = R(z_1)R(z_2) - I(z_1)I(z_2) \tag{5}$$

and

$$I(z_1z_2) = R(z_1)I(z_2) + I(z_1)R(z_2). \tag{6}$$

According to relations (5) and (6) in equation system $AXB + CX^*D = M$, we can calculate matrix $R(X)$ and $I(X)$ separately and define the unknown matrix $X \in \mathbf{C}^{m\times n}$ as $X = R(X) + iI(X)$. Therefore, we have

$$\begin{cases} R(AX) = R(A)R(X) - I(A)I(X) \\ I(AX) = R(A)I(X) + I(A)R(X) \end{cases}$$

and

$$\begin{aligned} R(AXB) &= R(AX)R(B) - I(AX)I(B) \\ &= R(A)R(X)R(B) - I(A)I(X)R(B) - R(A)I(X)I(B) - I(A)R(X)I(B) \end{aligned}$$

and in a similar manner

$$I(AXB) = R(A)R(X)I(B) - I(A)I(X)I(B) + R(A)I(X)R(B) + I(A)R(X)R(B)$$

By defining the Kronecker product of matrix, we can write the equation AXB as follows:

$$\begin{aligned} &((I(B)^T \otimes R(A)) + (R(B)^T \otimes I(A)))Vec(R(X)) \\ &-((I(B)^T \otimes I(A)) + (R(B)^T \otimes R(A)))Vec(I(X)). \end{aligned}$$

According to Lemma 2.2 from [3], for $C_1, X^T, D_1 \in \mathbf{R}^{m\times n}$, we have

$$Vec(C_1X^TD_1) = (D_1^T \otimes C_1)P(n,n)Vec(X),$$

in which $P(n,n)$ is a permutation matrix that causes some columns in $D_1^T \otimes C_1$ to be relocated. Therefore, the equation $C_1X^*D_1$ can be rewritten as the following real equation:

$$\begin{array}{l}((I(D)^T \otimes R(C)) + (R(D)^T \otimes I(C)))P(n,n)Vec(R(X)) \\ -((I(D)^T \otimes I(C)) + (R(D)^T \otimes R(C)))P(n,n)Vec(I(X))\end{array}.$$

With the new denomination, we have

$$\begin{cases}(R(B)^T \otimes R(A)) + (I(B)^T \otimes I(A)) = K_1 \\ (R(B)^T \otimes I(A)) + (I(B)^T \otimes R(A)) = K_2 \\ ((R(D)^T \otimes R(C)) + (I(D)^T \otimes I(C)))P(n,n) = K_3 \\ ((R(D)^T \otimes I(C)) + (I(D)^T \otimes R(C)))P(n,n) = K_4 \\ Vec(R(M)) = M_1 \text{ and } Vec(I(M)) = M_2\end{cases}.$$

So,

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}\begin{bmatrix} Vec(R(X)) \\ Vec(I(X)) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},$$

which is a real matrix equation which can easily be solved. If the equation system is like

$$\begin{array}{c}A_1XB_1 + C_1X^*D_1 = M_1 \\ A_2XB_2 + C_2X^*D_2 = M_2\end{array},$$

then we have

$$\begin{cases}(R(B_i)^T \otimes R(A_i)) + (I(B_i)^T \otimes I(A_i)) = K_{1i} \\ (R(B_i)^T \otimes I(A_i)) + (I(B_i)^T \otimes R(A_i)) = K_{2i} \\ ((R(D_i)^T \otimes R(C_i)) + (I(D_i)^T \otimes I(C_i)))P(n,n) = K_{3i} \\ ((R(D_i)^T \otimes I(C_i)) + (I(D_i)^T \otimes R(C_i)))P(n,n) = K_{4i} \\ Vec(R(M_i)) = M_{1i}, \quad Vec(I(M_i)) = M_{2i}\end{cases}$$

So,

$$\begin{bmatrix} K_{11} & K_{21} \\ K_{12} & K_{22} \\ K_{31} & K_{41} \\ K_{32} & K_{42} \end{bmatrix}\begin{bmatrix} Vec(R(X)) \\ Vec(I(X)) \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{12} \\ M_{21} \\ M_{22} \end{bmatrix}.$$

Considering the mentioned process, we can rewrite the equation $AX_1B + CX_2D = M$

as follows:

$$\begin{cases} R(AX_1) = R(A)R(X_1) - I(A)I(X_1) \\ I(AX_1) = R(A)I(X_1) + I(A)R(X_1) \end{cases},$$

$$\begin{aligned} R(AX_1B) &= R(AX_1)R(B) - I(AX_1)I(B) \\ &= R(A)R(X_1)R(B) - I(A)I(X_1)R(B) - R(A)I(X_1)I(B) - I(A)R(X_1)I(B), \\ I(AX_1B) &= R(A)R(X_1)I(B) - I(A)I(X_1)I(B) + R(A)I(X_1)R(B) + I(A)R(X_1)R(B), \end{aligned}$$

and similarly,

$$\begin{aligned} R(CX_2D) &= R(C)R(X_2)R(D) - I(C)I(X_2)R(D) - R(C)I(X_2)I(D) - I(C)R(X_2)I(D), \\ I(CX_2D) &= R(C)R(X_2)I(D) - I(C)I(X_2)I(D) + R(C)I(X_2)R(D) + I(C)R(X_2)R(D). \end{aligned}$$

Therefore, we have

$$\begin{bmatrix} (R(B)^T \otimes R(A)) - (I(B)^T \otimes I(A)) & -(R(B)^T \otimes I(A)) - (I(B)^T \otimes R(A)) \\ (I(B)^T \otimes R(A)) + (R(B)^T \otimes I(A)) & -(I(B)^T \otimes I(A)) + (R(B)^T \otimes R(A)) \end{bmatrix} \begin{bmatrix} Vec(R(X_1)) \\ Vec(I(X_1)) \end{bmatrix}$$
$$+$$
$$\begin{bmatrix} (R(D)^T \otimes R(C)) - (I(D)^T \otimes I(C)) & -(R(D)^T \otimes I(C)) - (I(D)^T \otimes R(C)) \\ (I(D)^T \otimes R(C)) + (R(D)^T \otimes I(C)) & -(I(D)^T \otimes I(C)) + (R(D)^T \otimes R(C)) \end{bmatrix} \begin{bmatrix} Vec(R(X_2)) \\ Vec(I(X_2)) \end{bmatrix}$$
$$= \begin{bmatrix} Vec(R(M)) \\ Vec(I(M)) \end{bmatrix}.$$

So, we can also rewrite the coupled complex equation system

$$\begin{cases} A_1X_1B_1 + C_1X_2D_1 = F_1 \\ A_2X_1B_2 + C_2X_2D_2 = F_2 \end{cases}$$

as a real equation system. ■

3. Numerical examples

In the previous part, we turned the coupled equation systems with complex coefficients matrix and unknowns into real equation systems, based on the multiplication properties of complex numbers. In this part, we will solve some numerical examples to check the efficiency of this method.

***Example* 3.1** Solve system $AXB + CX^*D = M$, if

$$A = \begin{bmatrix} 1+i & 2-i \\ 3+2i & 4+5i \end{bmatrix}, \ B = \begin{bmatrix} 5+6i & 1-2i \\ 4+3i & 5-2i \end{bmatrix}, \ C = \begin{bmatrix} -1-i & 2-3i \\ 4-3i & 5+2i \end{bmatrix},$$

$$D = \begin{bmatrix} -5+2i & -1-i \\ 7-8i & 1+9i \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -309+225i & 298-174i \\ -628-176i & 474+430i \end{bmatrix}.$$

The solution to this system using the mentioned method is

$$X = \begin{bmatrix} 4+2i & 2-2i \\ 1+3i & 3+7i \end{bmatrix} \quad \text{and} \quad X^* = \begin{bmatrix} 4-2i & 1-3i \\ 2+2i & 3-7i \end{bmatrix}.$$

***Example* 3.2** Solve the system

$$\begin{cases} A_1XB_1 + C_1X^*D_1 = M_1, \\ A_2XB_2 + C_2X^*D_2 = M_2, \end{cases}$$

if A_1, B_1, C_1, D_1 are respectively A, B, C, D in the previous example and

$$A_2 = \begin{bmatrix} 2+i & 4-i \\ 5 & 6-4i \end{bmatrix},\ B_2 = \begin{bmatrix} 7+3i & 2-5i \\ -9 & i \end{bmatrix},\ C_2 = \begin{bmatrix} 2-i & 1+3i \\ 10+i & -2-i \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 2-i & 2+5i \\ -7i & 1+i \end{bmatrix},\ M_1 = \begin{bmatrix} -25i & 279+102i \\ 472+596i & 59+602i \end{bmatrix},\ M_2 = \begin{bmatrix} 60-445i & 49+198i \\ 65-466i & 253+139i \end{bmatrix}.$$

The solution to this system is

$$X = \begin{bmatrix} 5+3i & 8+4i \\ 2-7i & 4+2i \end{bmatrix} \quad \text{and} \quad X^* = \begin{bmatrix} 5-3i & 2+7i \\ 8-4i & 4-2i \end{bmatrix}.$$

***Example* 3.3** Solve the system

$$\begin{cases} A_1X_1B_1 + C_1X_2D_1 = M_1, \\ A_2X_1B_2 + C_2X_2D_2 = M_2, \end{cases}$$

in which $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$ are defined matrices in the previous example and

$$M_1 = \begin{bmatrix} -301+160i & 440-15i \\ -401-612i & 519+183i \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} -166-239i & -138+135i \\ -589-764i & 820+205i \end{bmatrix}.$$

The solution to this system is

$$X_1 = \begin{bmatrix} -2i & 1+5i \\ -3-2i & 2-i \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 3 & 7-6i \\ 8+2i & 2-3i \end{bmatrix}.$$

***Example* 3.4** Solve the system $AX_1B + CX_2D = M$, if

$$A = \begin{bmatrix} -2-2i & 2+2i & 1-i \\ -2-i & 0 & 1+3i \\ 5-10i & 5+3i & 2-i \end{bmatrix},\ B = \begin{bmatrix} 4-2i & 1+2i & 0 \\ 2-5i & 4i & 2-3i \\ 3+5i & 6i & 1+3i \end{bmatrix},$$

$$C = \begin{bmatrix} -1+i & -1+4i & 5i \\ i & 3+4i & 2+i \\ 2+3i & 3i & 5 \end{bmatrix}, \ D = \begin{bmatrix} 10i & 3+2i & -1-5i \\ 11-6i & 10+i & 12-3i \\ 2-2i & 2-i & 3i \end{bmatrix}$$

$$\text{and} \ \ M = \begin{bmatrix} 292+232i & 287+471i & 359-86i \\ 542+291i & 274+288i & 146+74i \\ 405-218i & 121+305i & -50-352i \end{bmatrix}.$$

The solution to this system is

$$X = \begin{bmatrix} 2+i & 3i & 2+2i \\ 0 & 5-i & 1+7i \\ 4-2i & 3+i & 5-4i \end{bmatrix}.$$

4. Conclusion

This paper proposed a practical new method for obtaining the solution to general coupled matrix equations. This method is practical since the results are obtained by solving a real equation system. Furthermore, the new problem can be solved easily with the help of several algorithms.

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