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A new approach for determining multi-objective optimal control of semilinear parabolic problems

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Abstract
In this paper, two approaches based on evolutionary algorithms are applied to solve a multi-objective optimal control problem governed by semilinear parabolic partial differential equations. In this approach, first, we change the problem into a measure-theoretical one, replace this with an equivalent infinite dimensional multi-objective nonlinear programming problem and apply approximating schemes. Finally, non-dominated sorting genetic algorithm and multi-objective particle swarm optimization are employed to obtain Pareto optimal solutions of the problem. Numerical examples are presented to show the efficiency of the given approach.

Keywords Multi-objective optimal control problem · Pareto solution · Evolutionary algorithm · Radon measure

Mathematics Subject Classification 90C29 · 49M27

1 Introduction

In real applications, optimization problems are often described by introducing several objective functions conflicting with each other. This leads to multi-objective or multicriterial optimization problems; see, e.g., Ehrgott (2005). In the area of control engineering, multi-objective optimization has been discussed by control engineers [see, e.g., Gambier and Bareddin (2007)]. These objectives often involve conflict situations of many criteria, such as control energy, tracking performance and robustness. A suitable introduction on the concepts of MOOCP may be found in Gambier and Jipp (2011). Also, one may find an overview on multi-objective optimization applications in control engineering in Liu et al. (2003). Over the years, some indirect and direct approaches have been presented to extract analytical and approximate Pareto solutions of MOOCP 's Yalcin Kaya and Maurer (2014) and El-Kady et al. (2003). But, these approaches are facing some difficulties. For instance, convexity of

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the objectives is a basic requirement which limits the scope of applications of such methods Maity and Maiti (2005).

Rubio Rubio (1986) applied the embedding method for solving a control system governed by an elliptic equation to find the global control for the described system. The history of these ideas can be find for instance in Rubio (1990). Based on these papers, here we modify this method. In this manner, we present the problem in a variational form; next, it is transferred into a new theoretical measure problem in which one unknown positive Radon measure in a space of measures is sought. Then, a two-stage approximation is used to convert the optimal control problem to a finite dimensional NLP. The solution of this NLP is used to construct an approximate solution to the original multi-objective control problem. The proposed approach is practical and accurate enough and its accuracy can be improved as far as desired [see Fakharzadeh et al. (2013)]. Due to outstanding abilities of evolutionary algorithms in finding Pareto solutions of multi-objective optimization problems, two evolutionary algorithms, MOPSO and NSGAII, are employed to find a Pareto optimal control for multi-objective optimal nonlinear problem Borzabadi et al. (2016) and Kumar and Minz (2014).

The paper is organized as follows. In Sect. 2, the multi-objective optimal control problem is formulated and transformed into a multi-objective nonlinear programming problem which is considered in Sect. 3. In Sect. 4 the algorithm for solving the problem has been presented. The numerical strategy and results are discussed in Sect. 5.

2 Problem formulation

Let $\Omega \subset R^d$, $d \in \{1, 2, 3\}$, be an open and bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. For given $T > 0$, we set $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$. Then, we consider the following class of multi-objective optimal control problems governed by semilinear parabolic equations Iapichino and Volkwein (2015):

$$\min \quad \mathcal{J}(y,u) = \begin{pmatrix} \mathcal{J}_1(y,u) \\ \mathcal{J}_2(y,u) \\ \mathcal{J}_3(y,u) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\int_\Omega |y(T,.) - y_\Omega|^2 \mathrm{d}\mathbf{x} \\ \frac{1}{2}\int_0^T \int_\Omega |y - y_Q|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \\ \frac{1}{2}\sum_{i=1}^m |u_i - u_i^d|^2 \end{pmatrix} \tag{1}$$

subject to the semilinear parabolic differential problem

$$\begin{aligned} &y_t(t,\mathbf{x}) - \Delta y(t,\mathbf{x}) + y^3(t,\mathbf{x}) = \sum_{i=1}^m u_i b_i(\mathbf{x}) + f(t,\mathbf{x}) \ \text{ for } \ (t,\mathbf{x}) \in Q, \\ &\frac{\partial y}{\partial n}(t,s) = 0 \ \text{ for } \ (t,s) \in \Sigma, \quad y(0,\mathbf{x}) = y_0(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Omega \end{aligned} \tag{2}$$

and to the bilateral control constraints

$$u \in \mathcal{U}_{ad} = \{\tilde{u} = (\tilde{u_1}, ..., \tilde{u_m}) \in R^m | u_i^a \le \tilde{u_i} \le u_i^b \ \ for \ \ 1 \le i \le m\}. \tag{3}$$

In (1), we assume that $y_\Omega \in L^\infty(\Omega)$, $y_Q \in L^\infty(Q)$, $u^d = (u_1^d, ..., u_m^d)^T \in R^m$. Furthermore, we suppose that $b_1, ..., b_m \in L^\infty(\Omega)$, $y_0 \in L^\infty(\Omega)$. In (3), let $u_i^a, u_i^b \in R$ satisfying $u_i^a \le u_i^b$ for $1 \le i \le m$.

In general, it is difficult to identify a classical solution for this problem; thus, it has usually been tried to find a weak solution for it Munch (2009). So, we change the problem into the variational form as follows.

Impressing every $\varphi \in H_0^1(\Omega \times (0,T))$, the mentioned constraint in (2) and integrating this formula can be represented as:

$$\begin{aligned}&\int_0^T \int_\Omega (y_t(t,\mathbf{x}) - \Delta y(t,\mathbf{x}) + y^3(t,\mathbf{x}))\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \int_0^T \int_\Omega \left(\sum_{i=1}^m u_i b_i(\mathbf{x}) + f(t,\mathbf{x})\right)\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t\end{aligned} \tag{4}$$

According to Green's inequality

$$\int_\Omega \Delta y(t,\mathbf{x})\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x} - \int_\Omega y(t,\mathbf{x})\Delta\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Sigma \left(\frac{\partial y}{\partial n}\varphi - \frac{\partial \varphi}{\partial n}y\right)\mathrm{d}S$$

and by using initial conditions, we have

$$\int_\Omega \Delta y(t,\mathbf{x})\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega y(t,\mathbf{x})\Delta\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}$$

then, equation (4) can be rewritten as:

$$\begin{aligned}&\int_0^T \int_\Omega (y_t(t,\mathbf{x}) + y^3(t,\mathbf{x}))\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^T \int_\Omega y(t,\mathbf{x})\Delta\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t \\ &= \int_0^T \int_\Omega \left(\sum_{i=1}^m u_i b_i(\mathbf{x}) + f(t,\mathbf{x})\right)\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t\end{aligned} \tag{5}$$

Because the objective function $\mathcal{J}_1(y,u)$ is independent of time, we can write it as follows:

$$\frac{1}{2T}\int_0^T \int_\Omega |y(T,.) - y_\Omega|^2 \mathrm{d}\mathbf{x}\mathrm{d}t$$

Thus, one can state Problem (1) in this form:

$$\begin{aligned}&\min \quad \mathcal{J}(y,u) = \begin{pmatrix} \frac{1}{2T}\int_0^T \int_\Omega |y(T,.) - y_\Omega|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \\ \frac{1}{2}\int_0^T \int_\Omega |y - y_Q|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \\ \frac{1}{2}\sum_{i=1}^m |u_i - u_i^d|^2 \end{pmatrix} \\ &s.t. \int_0^T \int_\Omega (y_t(t,\mathbf{x}) + y^3(t,\mathbf{x}))\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^T \int_\Omega y(t,\mathbf{x})\Delta\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t = \\ &\int_0^T \int_\Omega \left(\sum_{i=1}^m u_i b_i(\mathbf{x}) + f(t,\mathbf{x})\right)\varphi(t,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned} \tag{6}$$

To solve (6), we change the problem and consider a new one with a different formulation.

3 Embedding the solution space

The solution method which is based on an embedding process involves several stages to set up a nonlinear multi-objective programming problem whose solution converges to the solution of the original problem [see Rubio (1986)]. This is one of the outstanding advantages of this method. First, a measure theoretical approach and a two-stage approximation are used to convert the optimal control problem to a finite dimensional NLP. The solution of this NLP is used to construct an approximate solution to the original control problem. The proposed

approach is practical and accurate enough and its accuracy can be improved as far as desired [see Fakharzadeh et al. (2013)].

By considering function y as the trajectory and (u_i, y_t) as the control vector, problem (6) will be a multi-objective control problem. For solving the problem and expressing its method, we need to present the following definition. In this manner, suppose that the domain of control functions y_t and u_i are represented by $\dot{Y}$ and U_i, respectively.

Definition 3.1 vector $p = (y, y_t, u_i)$ is called admissible when it satisfies the following conditions:

1. The control functions (y_t, u_i) are bounded and continuous and take their values on compact sets $\dot{Y}$ and $U_i \subset R$;
2. y is the bounded solution of the semilinear parabolic system (2);

The set of all admissible vectors are denoted by $\mathcal{F}$. If the system is controllable, set $\mathcal{F}$ is non-empty (this can be seen in Rubio (1986), for instance).

We define $D = [0, T] \times X_1 \times X_2 \times Y \times \dot{Y} \times Y_T$ (the specific domain for variable $t, x_1, x_2, y, y_t, y(T)$, respectively). In general, finding admissible vector p may be difficult or estimating its numerical value may not be possible. If set $\mathcal{F}$ gets bigger through a method, these problems will be solved. The basis of this change is on the fact that there is a one-to-one correspondence between admissible vectors and a set of positive Radon measures. Therefore, an admissible vector $(t, x_1, x_2, y, y_t, y(T)) \in \mathcal{F}$ introduces the following linear, positive and bounded functional $\Gamma_p(F)$ on $C(D)$ as:

$$\Gamma_p(F) = \int_0^T \int_\Omega F(t, x_1, x_2, y, y_t, y(T)) d\mathbf{x} dt \quad \forall F \in C(D); \tag{7}$$

Problem (6) can be expressed on the basis of functional Γ_p for each $p \in \mathcal{F}$.

$$\begin{aligned} \min \quad & \mathcal{J}(y, u) = \begin{pmatrix} \frac{1}{2T}\Gamma_p(|y(T, .) - y_\Omega|^2) \\ \frac{1}{2}\Gamma_p(|y - y_Q|^2) \\ \frac{1}{2}\sum_{i=1}^m |u_i - u_i^d|^2 \end{pmatrix} \\ & s.t. \ \Gamma_p((y_t(t, \mathbf{x}) + y^3(t, \mathbf{x}))\varphi(t, \mathbf{x})) - \Gamma_p(y(t, \mathbf{x})\Delta\varphi(t, \mathbf{x})) \\ & = \Gamma_p(\textstyle\sum_{i=1}^m u_i b_i(\mathbf{x}) + f(t, \mathbf{x}))\varphi(t, \mathbf{x})). \end{aligned} \tag{8}$$

Based on Riesz representation theorem (Rudin 1983), there is a unique measure μ_p on $C(D)$ corresponding to bounded and linear functional Γ_p, so that:

$$\Gamma_p(F) = \int_D F d\mu_p \equiv \mu_p(F), \quad \forall F \in C(D);$$

Therefore, one can transfer problem (8) into a measure space by

$$(t, x_1, x_2, y, y_t, y(T)) \in \mathcal{F} \longmapsto \mu_p \in \mathcal{M}^+(D).$$

where $\mathcal{M}^+(X)$ is the set of all positive Radon measures on X. So, we have

$$\begin{aligned}
&\min \quad \mathcal{J}(y,u) = \begin{pmatrix} \frac{1}{2T}\mu_p(|y(T,.) - y_\Omega|^2) \\ \frac{1}{2}\mu_p(|y - y_Q|^2) \\ \frac{1}{2}\sum_{i=1}^{m} |u_i - u_i^d|^2 \end{pmatrix} \\
&s.t.\ \mu_p((y_t(t,\mathbf{x}) + y^3(t,\mathbf{x}))\varphi(t,\mathbf{x})) - \mu_p(y(t,\mathbf{x})\Delta\varphi(t,\mathbf{x})) \\
&= \mu_p(\sum_{i=1}^{m} u_i b_i(\mathbf{x}) + f(t,\mathbf{x}))\varphi(t,\mathbf{x})).
\end{aligned} \tag{9}$$

It was proved by Rubio (1986) that such a transformation is a one-to-one mapping. To achieve something new, we enlarge the underlying space and consider the problem of finding a minimizer of measure, say μ^*, on the space of all positive related Radon measures which are just satisfied to the conditions of (9) and seek to minimize functionals over this new and larger set called W (not only those that are induced from Riesz Representation theorem); therefore, our method is somehow global.

The space $\mathcal{M}^+(D)$ is a linear space which will become a locally convex topological vector space when it gives the weak*-topology.

Even though (9) has an optimal solution Rubio (1986), it is still very difficult to obtain the exact solution since the underlying spaces are not finite-dimensional, the number of equations is not finite and the unknowns are measures. Therefore, it is completely acceptable to seek for a suboptimal solution. Thus, first, by choosing suitable dense subsets in the appropriate spaces and then, by choosing a finite number of them, the problem is approximated by a semi-finite nonlinear programming one.

3.1 Identifying a nearly optimal solution

It is possible to approximate the solution of problem (9) by the solution of a finite-dimensional nonlinear one of sufficiently large dimensions. Besides, by increasing the dimension of the problem, the accuracy of the approximation can be increased. In the first estimation step, the problem is turned into a semi-finite non-linear programming one. This will be achieved by choosing countable sets of functions whose linear combinations are dense in appropriate spaces and then by selecting a finite number of constraints. Let $\{\varphi_j : \ j \in N\}$ be countable dense (in the topological convergence sense) sets in space $H_0^1(\Omega \times (0,T))$. By choosing a finite number of functions in this set, the solution of (9) can be approximated by the solution of the following one:

$$\begin{aligned}
&\min \quad \mathcal{J}(y,u) = \begin{pmatrix} \frac{1}{2T}\mu_p\left(|y(T,.) - y_\Omega|^2\right) \\ \frac{1}{2}\mu_p\left(|y - y_Q|^2\right) \\ \frac{1}{2}\sum_{i=1}^{m} |u_i - u_i^d|^2 \end{pmatrix} \\
&s.t.\ \mu_p((y_t(t,\mathbf{x}) + y^3(t,\mathbf{x}))\varphi_j(t,\mathbf{x})) - \mu_p(y(t,\mathbf{x})\Delta\varphi_j(t,\mathbf{x})) \\
&= \mu_p\left(\sum_{i=1}^{m} u_i b_i(\mathbf{x}) + f(t,\mathbf{x})\right)\varphi_j(t,\mathbf{x})), \quad j = 1, ..., M_1.
\end{aligned} \tag{10}$$

Because of the density property of the selected sets in (10), its solution tends to the solution of (9) when $M_1 \to \infty$; thus, if number M_1 is large enough, (10) is a good approximation of

our main problem. Now, the number of constraints of the problem is finite but the problem is still infinite since the underlying space is a subspace of measures.

Rosenbloom Rosenbloom (1956) established the basic theory underlying the solution of problem such as (10); in particular, he showed that if set W is nonempty, the infimum is attained at a measure which is a positive combination of finite number measures in $\mathcal{M}^+(D)$ which are primary, that is, it takes only values 0 and 1 Alimorad and Fakharzadeh (2017). From Proposition III.2 in Rubio (1986), μ^* has the form $\mu^* = \sum_{n=1}^{N} \alpha_n^* \delta(q_n^*)$, where $q_n^* \in D$ and $\alpha_n^* \geq 0$ for $n = 1, 2, ..., N$ [here $\delta(q)$ is a unitary atomic measure, characterized by $\delta(q)F = F(q)$ for $F \in C(D)$ (see Fakharzadeh and Rubio 1999)]. Thus, the measure-theoretical optimization problem is equivalent to a nonlinear optimization problem in which the unknowns are coefficients α_n^* and supports $\{q_n^*\}$. It would be much more convenient if we could minimize the functions only with respect to the coefficients α_n^*. However, we do not know the supports of the optimal measure. This is possible by suitable discretization of space D into N equal cells (big enough) and then, selection of points $\{q_n^*\}$, $n = 1, 2, .., N$ from this cells. Let D'_Ω be a countable dense subset of D; then, as a consequence of Proposition III.3 in Rubio (1986), measure $\mu \in \mathcal{M}^+(D)$ of the form $\sum_{n=1}^{N} \alpha_n \delta(q_n)$ exists such that $q_n \in D'_\Omega$. This leads us to discretize D by nodes $q_n = (t_n, x_{1_n}, x_{2_n}, y_n, \dot{y_n}, y_{T_n})$, $n = 1, 2, .., N$ lying in D'_Ω and solution of problem (10) is obtained by following problem:

$$\begin{aligned} \min: \quad \mathcal{J}(\alpha, u) = \min & \begin{pmatrix} \frac{1}{2T} \sum_{n=1}^{n=N} \left(|y_n(T, .) - y_{\Omega_n}|^2 \right) \\ \frac{1}{2} \sum_{n=1}^{n=N} \left(|y_n - y_{Q_n}|^2 \right) \\ \frac{1}{2} \sum_{i=1}^{m} |u_i - u_i^d|^2 \end{pmatrix} \\ s.t. \sum_{n=1}^{n=N} \left(\left(y_{t_n} + y_n^3 \right) \varphi_j \right) - \sum_{n=1}^{n=N} (y_n \Delta \varphi_j) & \\ = \sum_{n=1}^{n=N} \left(\sum_{i=1}^{m} u_i b_i(\mathbf{x}) + f(t_n, \mathbf{x}) \varphi_j \right), \quad j = 1, ..., M_1. & \end{aligned} \tag{11}$$

Problem (11) is still non-linear because u_i are unknown. Now, using two evolutionary algorithms MOPSO and NSGAII for (11), the optimal coefficients $(\alpha_1^*, ..., \alpha_N^*)$ and u_i^* would be found as explained in the next section.

It is worth mentioning that Galerkin method also expresses an integral problem as a linear problem. In the method we put forth in this paper, for identifying the unknown function, support points of functions domain are used. On the other hand, there is no need to define polynomial functions which are linearly independent. It is worth reminding that by an unsuitable definition of polynomial functions in Galerkin method, it is likely that the problem is unsolvable (Introduction to Galerkin Methods 2016).

4 Algorithm of the approach

To apply the mentioned method for solving Problem (11) practically, here we present an algorithmic path for the solution procedure.

Initialization step:

I: The given sets $[0, T]$, X_1, X_2, Y, $\dot{Y}$, Y_T (the specific domain for variable t, x_1, x_2, y, y_t, $y(T)$, respectively), which from D are divided into n_1, n_2, n_3, n_4, n_5 and n_6 equal parts;

so that, the $N = n_1.n_2.n_3.n_4.n_5.n_6$, the number of 6-dimensional cells in the related spaces is obtained. Then, in each of these 6-dimensional cells, arbitrary points $q_n = (t_n, x_{1_n}, x_{2_n}, y_n, \dot{y_n}, y_{T_n})$ are selected.
II: For fixed numbers M_1, we select M_1 number of φ_i functions. Now, one is able to set up the finite nonlinear programming problem (11) with $N + m$ variables and M_1 constraints, which is dependent on the variables u_i.

We have proved in Alimorad and Fakharzadeh (2017) the convergence of the above-mentioned method according to three propositions.

To solve Problem (11) with the help of evolutionary algorithms, first, we consider the required concepts.

Definition 4.1 (Dominates) Given the vector of objective functions $(E_1(x, u), ..., E_k(x, u))$; we say that candidate (x_1, u_1) dominates (x_2, u_2), (and denote it as $(x_1, u_1) \prec (x_2, u_2)$), if for each $i \in \{1, ..., k\}$, $E_i(x_1, u_1) \leq E_i(x_2, u_2)$ and for some $i \in \{1, ..., k\}$, $E_i(x_1, u_1) < E_i(x_2, u_2)$.

Definition 4.2 (Pareto Optimal): The pair $(\bar{x}, \bar{u}) \in P$ is said to be a Pareto Optimal solution or non-dominated solution if and only if there is not any admissible pair which dominates it.

4.1 NSGAII and MOPSO algorithm steps

An algorithm based on the previous discussions is summarized in this subsection (see Borzabadi et al. (2016) for more details):

The objective of the NSGA algorithm is to improve the adaptive fit of a population of candidate solutions to a Pareto front constrained by a set of objective functions. The algorithm uses an evolutionary process with surrogates for evolutionary operators including selection, genetic crossover, and genetic mutation. The population is sorted into a hierarchy of sub-populations based on the ordering of Pareto dominance. A similarity between members of each sub-group is evaluated on the Pareto front, and the resulting groups and similarity measures are used to promote a diverse front of non-dominated solutions Borzabadi et al. (2016). Non-dominated Sorting Genetic Algorithm (NSGAII) steps are

1. Choose a population of random individuals as $(\alpha_{i0}, ..., \alpha_{in}, u_{i0}, ..., u_{in})$.
2. Evaluate objective functions values for each individual.
3. Assign ranks based on Pareto dominance.
4. Calculate the crowding distance for each individual.
5. Sort the population in a descending manner.
6. Apply the rules of generating new population (such as crossover).
7. Obtain Pareto set.
8. Repeat the main step for a predetermined number of iterations.

In PSO, the manipulation of a swarm is different from the evolutionary algorithms, because it promotes a cooperative model rather than a competitive one. An adaptable velocity vector is used by PSO, which changes particle position at each iteration of the algorithm. It exploits information springing from its own previous experiences to move toward the promising regions of the search space Kumar and Minz (2014). Multi-objective particle swarm optimization (MOPSO) steps are

1. Choose a population of random individuals as $(\alpha_{i0}, ..., \alpha_{in}, u_{i0}, ..., u_{in})$.
2. Evaluate objective function values for each individual.
3. Assign ranks based on Pareto dominance.

4. Create grid search space.
5. Select a leader for any particle and update its position.
6. Update the best previous position for any particle.
7. Add a non-dominated population to repository.
8. Delete dominated solutions from archive.
9. Delete extra particles from archive and re-create grid search space.
10. Repeat the main step for a predetermined number of iterations.

Although evolutionary algorithms produce robust solutions, they may not always be globally optimal. However, their solutions are robust most of the time. In real-life applications, this can be very valuable because in general, finding good solutions which are highly robust is very difficult. Multiple runs are required to obtain the Pareto-front by using a scalarization approach (In each run, a different weight value is considered). However, an evolutionary algorithm achieves the whole trade-off solutions in just a single run. That being so, choosing weights in scalarization approach is important in that they can influence the final solution Eichfelder (2009).

5 Numerical result

Now, to show the efficiency of our method and to explain how it works, we solve a numerical example with two evolutionary algorithms. It is worth mentioning that this example is taken from Iapichino and Volkwein (2015) as well as from other studies cited by it in order for the readers to be able to compare and contrast the two methods.

Example 5.1 In this example, we consider the initial conditions and region Ω as reference Iapichino and Volkwein (2015), to examine the new method numerically.

We consider (1) with spatial domain $\Omega = (0,1) \times (0,1) \subset R^2$, final time $T = 1$, desired states $y_\Omega = 0$, $y_Q(t, \mathbf{x}) = 100tcos(2x_1)cos(2x_2)$, initial condition $y_0(x) = 0$ and inhomogeneity $f(t, \mathbf{x}) = 10tx_1$. Furthermore, for $m = 4$ each shape function $b_i(x)$, $i = 1, ..., 4$, has the support in a quarter of the domain and $u^d = (0.5, -4, -0.5, 4)^T \in R^4$. Also, we chose $u_1 \in [-3, 3]$, $u_2 \in [-8, -1]$, $u_3 \in [-5, -2]$, $u_4 \in [-1, 6]$ (not optimal controls). To discretize $D = [0, T] \times X_1 \times X_2 \times Y \times \dot{Y} \times Y_T$, we choose $M = 10^3 \times 5^3$ points in these sets:

selecting 10 points in $[0, T]$ for t as:

$$0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, 1;$$

10 points in X_1 for x_1 as:

$$0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, 1;$$

5 points in X_2 for x_2 as:

$$0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1;$$

5 values for y as:

$$0, 0.375, 0.75, 1.125, 1.5;$$

10 values for y_t as:

$$0, 0.17, 0.34, 0.51, 0.68, 0.85, 1.02, 1.19, 1.36, 1.5;$$

5 values in Y_T for $y(T)$ as:

$$0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1.$$

Then, M number of nodes $(t, x_1, x_2, y, y_t, y_T) \in D$ is introduced and hence all nodes belong to the dense subset of D).

To set up the linear programming problem (11), we consider $\varphi_j(t, \mathbf{x}) = x_1 x_2 (x_1 - 1)(x_2 - 1)t \in H_0^1(\Omega \times [0, T])$ for $j = 1, 2, 3$. Thus, nonlinear programming problem (11) having one constraint and $M = 10^3 \times 5^3$ variables is set up and solved by NSGAII and MOPSO algorithms and the optimal points (u_i^*) and $\alpha_n^* > 0$ are obtained. The optimal

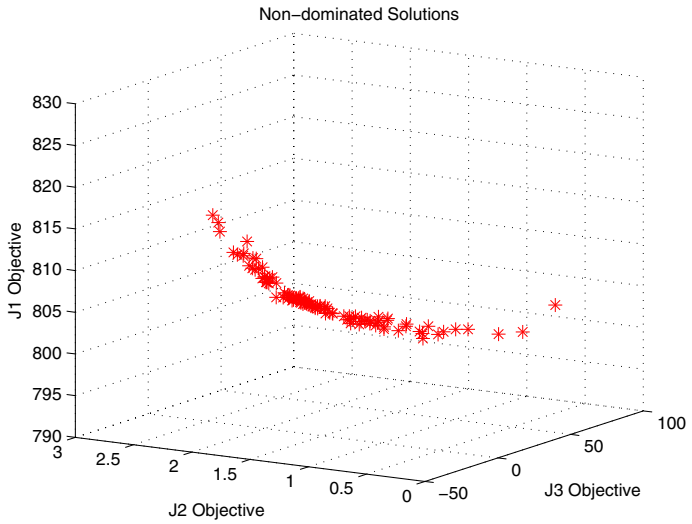

Fig. 1 Optimal objective functions with NSGAII algorithm

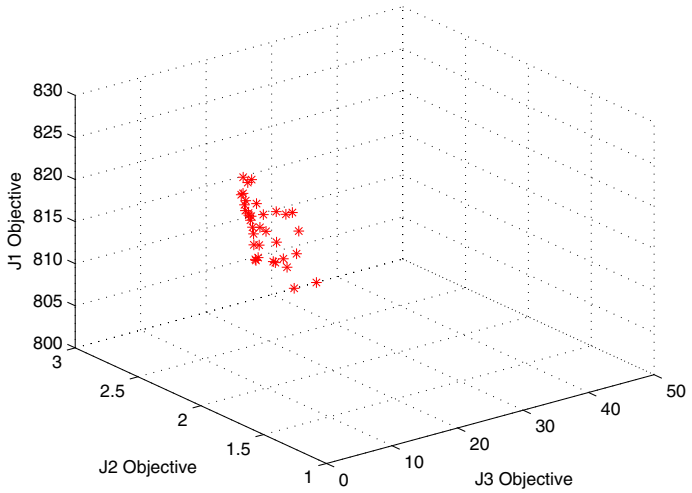

Fig. 2 Optimal objective functions with MOPSO algorithm by 10 iterations

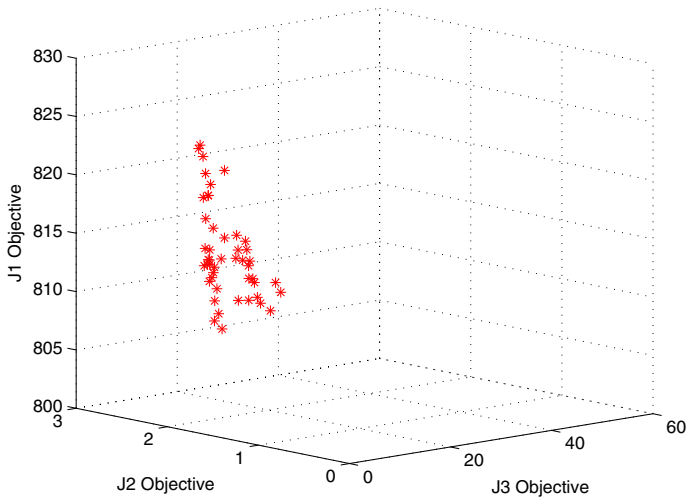

Fig. 3 Optimal objective functions with MOPSO algorithm by 50 iterations

values of the controls, which are obtained from NSGAII algorithm, are $u_1 = 0.1512$, $u_2 = -3.8512$, $u_3 = -2.3967$, $u_4 = 3.3548$, and from MOPSO algorithm, are $u_1 = -1.8405$, $u_2 = -4.3689$, $u_3 = -2.0000$, $u_4 = 4.0000$, respectively.

The objective functions are represented in Figs. 1, 2 and 3.

By comparing the results with reference Iapichino and Volkwein (2015), the ability of the proposed method based on the measures for solving multi-objective nonlinear optimization control problems is characterized. In this method, we can rewrite the problem of optimal control as a multi-objective nonlinear programming problem. The new problem can be solved easily with the help of innovative algorithms. This method is simpler because it is independent of the initial value. Moreover, the cpu time is about 0.6 seconds. While in reference Iapichino and Volkwein (2015) (p. 7), using the Reduced-order method, the time is about 7.3 for solving this problem.

6 Conclusion

This paper proposed a practical approach for obtaining the solution to general multi-objective optimal control of semilinear parabolic problems. Compared with other methods, this approach is more practical since the results are obtained by solving one NLP and it is independent of the initial value. Furthermore, the new problem can be solved easily with the help of innovative algorithms. Besides being easier, it takes less time. It is also especially practical and accurate enough for systems with nonlinear terms. Also, after approximating the problem, the more the number of variables and constraints is, the more precise the problem will be.

References

Alimorad H, Fakharzadeh JA (2017) A theoretical measure technique for determining 3D symmetric nearly optimal shapes with a given center of mass. Comput Math Math Phys 57(7):12251240

Borzabadi AH, Hasanabadi M, Sadjadi N (2016) Approximate Pareto optimal solutions of multi-objective optimal control problems by evolutionary algorithms. Control Optim Appl Math 1(1):1–19
Ehrgott M (2005) Multicriteria optimization. Springer, Berlin
Eichfelder G (2009) An adaptive scalarization method in multi-objective optimization. SIAM J Optim 19(4):1694–1718
El-Kady MM, Salim MS, El-Sagheer AM (2003) Numerical treatment of multi-objective optimal control problems. Automatica 39:4755
Fakharzadeh JA, Rubio JE (1999) Shape and measure. J Math Control Inf 16:207–220
Fakharzadeh JA, Alimorad H, Rafiei Z (2013) Using linearization and penalty approach to solve optimal shape design problem with an obstacle. J Math Comput Sci 7:43–53
Gambier A, Bareddin E (2007) Multi-objective optimal control: an overview. In: IEEE conference on control applications, CCA, Singapore, pp 170–175
Gambier A, Jipp M (2011) Multi-objective optimal control: an introduction. In: Proceedings of the 8th Asian control conference (ASCC11). Kaohsiung, Taiwan, pp 15–18
Introduction to Galerkin Methods (2016) http://fischerp.cs.illinois.edu/tam470/refs/galerkin2.pdf. Accessed October 2018
Iapichino SL, Volkwein ST (2015) Reduced-order multi-objective optimal control of semilinear parabolic problems. Konstanzer Online-Pablikations Syst 347:1–8
Kumar V, Minz S (2014) Multi-objective particle swarm optimization: an introduction. Smart Comput Rev 4(5):335–353
Liu GP, Yang JB, Whidborne JF (2003) Multi-objective optimization and control. Research Studies Press Ltd., Exeter
Maity K, Maiti M (2005) Numerical approach of multi-objective optimal control problem in imprecise environment. Fuzzy Optim Decision Mak 4:313–330
Munch A (2009) Optimal internal dissipation of damped wave equation using a topological approach. Int J Appl Math Comput Sci 19:15–37
Rosenbloom PC (1956) Qudques classes de problems exteremaux. Bull Soc Math France 80:183–216
Rubio JE (1986) Control and optimization: the linear treatment of nonlinear problems. Manchester university Press, Manchester
Rubio JE (1990) Modern trends in calculus of variations and optimal control theory. In: Geise R (ed) Vortagsauszuge, Mathematiker kongress, 155–163, Berlin Mathematische Gesellschaft
Rudin W (1983) Real and complex analysis, 2nd edn. Tata McGraw-Hill Publishing Co Ltd., New Dehli
Yalcin Kaya C, Maurer H (2014) A numerical method for nonconvex multi-objective optimal control problems. Comput Optim Appl 57(3):685–702