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Original research article

A new approach for solving the obstacle problems in three-dimensions

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A B S T R A C T

The aim of this paper is to propose an approach to find an optimal surface with a given image in the cylindrical coordinate so that it overcomes the given obstacles. In addition, several circumstances, such as the symmetry or asymmetry of the surface are also considered. For this aim, the obstacle is described by variational inequalities and the problem is expressed as an optimal control problem by introducing artificial controls. Next, considering a variational presentation and applying an embedding procedure, the problem is transferred into one whose unknown is an optimal Radon measure. Hence, the shape design problem is changed into an infinite linear one whose solution is guaranteed. Then, for each given domain, two stages of approximation is used to find the optimal surface. Finally, the nearly optimal solution of original problem is constructed via a finite linear programming whose smoothness is improved by applying outlier detection and smooth fitting. Numerical examples are presented and the results are compared.

1. Introduction and backgrounds

The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. It is deeply related to the study of minimal surfaces and the capacity of a set in potential theory as well. Applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control and financial mathematics [1,2]. In these problems, the solution breaks down into a region where the solution is equal to the obstacle function, known as the contact set, and a region where the solution is above the obstacle; the interface between the two region is the free boundary [3]. To study further details, refer to [4] where the obstacle problem and free boundary problem are thoroughly reviewed.

Effort has been put into optimizing such problems by transforming them into shape optimization problems based on the idea put forward by Young (see [5]) and a famous method of this kind was theoretically established by Rubio [6]. This method has been extended and improved by others [7–9]. But, indeed, 3-D optimal shape design (OSD) methods are problematic as explained in [10], there is a considerable number of methods for designing two-dimensional optimal shapes (2-D OS); the level set method [11], topological method [12], adaptive wavelet collocation method [13], indirect shooting method [14], mapping method [15] and numerical methods based on finite element (FEM) and finite difference [16] are some instances.

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Regarding solving optimal control of obstacle problem using direct pseudo-spectral method [17], wavelet-based adaptive mesh refinement [18] and adaptive wavelet collocation method [19] and also references mentioned in those papers, it could be argued. Unfortunately, only few books and articles are available on three-dimensions (3-D) shape optimization, although some industrial factors cannot be implemented in a two-dimensions (2-D) manner and a 3-D design is needed.

For creating a 2-D architectural, we have to work out the scale and use old-fashioned techniques to ensure consistency and accuracy of measurements. 3-D modeling removes inconsistencies presenting an excellent level of accuracy. Additionally, 3-D printing is based on 3-D modeling. Manufacturers will first create 3-D models on a computer. This design will then be sent to the 3-D printer as a result of the final product which created. Also, Architects can now create complete structures using 3-D modeling programs. In this vein, there is no need for heaps of drawings and materials — simply one 3-D modeling file that showcases every aspect of the building. 3-D modeling is a vital part of the modern filmmaking process too. It has also, played an important role in advancing video game graphics.

Moreover, 3-D modeling has helped improve healthcare and medical science significantly and is used in the pharmaceutical industry in a variety of different ways. It can also be used to visualize the human body. Over time, medical students would look at such things as plastic skeletons and model organs. In this regard, this modeling can be used to help create artificial organs and medical equipment. For instance, researchers can use modeling programs to create new equipment designs — scalpels and forceps. Further, chemists can use 3D modeling to look at complex chemical structures of different elements and atoms. They can use it to display practically anything; That is, things that may not be visible by the naked eye.

In this paper, we are going to present a developed version of the works done in [20] for 2-D and in [10] for 3-D space, to determine a major class of optimal surfaces for obstacle problems. Indeed, our main purpose is to present a linearization method to solve the obstacle problem in a 3-D originally. In general, the main steps presented in this paper could be introduced in this way: First, we state the classical problems as integral equations. Then, by introducing positive Radon measures, we change the problem into measure spaces. In the third step, to overcome problems such as solution existence, we extend the space. In this step, we have an in infinite linear programming problem (LPP) in measure spaces. Finally, after solving the LPP, path functions and optimal control are calculated. So far, this method has been presented in one and two-dimensional states and compared to other methods, it has the main advantages such as solution existence, the new problem's being linear despite the original problem's being non-linear, and finding a general solution for the problem.

In this paper, this method has been extended for solving the optimal shape design problem in 3-D. Using this method, we obtain the 3-D unknown surface directly. Some of the advantages of this method are: computer programming for this method is easy in that the problem is approximated by a finite LPP. This method is independent of the prime shape and its run time is short. Furthermore, it is a general method which is not dependent on the problem type. In other words, it does not require a specific basic information such as objective function derivative and constraints. It is possible to make the obtained surface smooth by identifying outlier points and then, by curve fitting in the mentioned method.

The paper is organized as follows: the next section is devoted to the statement of the problem. The aim of Section 3 is to recast the problem into variational form. Section 4 is devoted to approximation schemes. In Section 5, we discuss outlier detection for smoothness. In Section 6, we investigate the convergence of the proposed numerical approach. Some numerical simulations are done in Section 7 and we conclude with some remarks in Section 8.

2. Problem statement in a general form

In free boundary problems, it is often needed to find the boundary of some domains in 3-D space; the obstacle kind of these problems can be characterized by a variational inequality, as we will explain while introducing the problem. Based on this fact, the main goal, in this section, is to extend the mentioned method in [7] for solving obstacle problems in 3-D space.

Here we are looking for an unknown shape which is bounded and has a specified volume that is placed on top of (r, θ) plane; its boundary includes the unknown surface S with supposed equation $z = f(r, \theta)$. Due to the smoothness and continuous nature of surface S, we assume that function $z = f(\theta, r)$ is an absolutely continuous function. Also, its image in the plane (r, θ) is the given region D, which is a smooth and simple closed curve. This surface passes through a specified point, say (θ_0, r_0, z_0), that $(\theta_0, r_0) \in \partial D$, with a height bounded between z_{min} and z_{max} (Understand the issue, we have come up with a hypothetical figure (i.e., Fig. 1)). The set of all admissible surfaces (or any unknown mentioned shape C) is introduced as:

$$S_A = \{z = f(r, \theta) | \partial C = S \cup D, \int_C dV = L, (\theta_0, r_0, z_0) \in S,\ (\theta_0, r_0) \in \partial D, S \geq \Psi \ \ in\ \Omega_0\},$$

where the region $D \subset R^2$ and its boundary ∂D are defined as follows ($h(\theta)$ is a given continuous function):

$$D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq h(\theta)\}, \quad \partial D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, r = h(\theta)\}.$$

Also, the relation $S \geq \Psi(r, \theta, z),\ \ \forall (r, \theta, z) \in \Omega_0$ means that the surface S has to dominate the obstacle Ψ in the region shown by Ω_0; it is necessary to note that Ψ has the property that the obstacle condition could be shown by a variational inequality, or considered in the discretization or bound conditions schemes of variables or both. The goal of the obstacle problem is to find an admissible surface $S \in S_A$ such that besides overcoming the given obstacle, it is minimizing the specified integral performance over S. So, for the given D, Ω, Ω_0 and Ψ, the problem can be classified as:

$$\begin{aligned} &Min_{S \in S_A} \ \ I(S) = \int_S f_0 d\sigma \\ &S.\ to:\ \ z_{min} \leq z \leq z_{max}. \end{aligned} \tag{1}$$

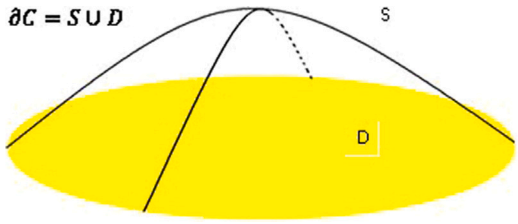

Fig. 1. A general unknown shape.

The integrand f_0 is supposed to be integrable (or more precisely, differentiable where $d\sigma$ is the differential surface area) on a specified region Ω; for instance, if $f_0(r,\theta,z)=1$, then we are faced with the obstacle minimum surface problem [2].

We remind that, the condition $(\theta_0, r_0, z_0) \in S$ can be written as $f(\theta_0, r_0) = z_0$. Moreover, because the shadow of surfaces is the simple closed curve ∂D in polar plane and $(\theta_0, r_0) \in \partial D$, we have $f(\theta_0, r_0) = f(\theta_0 + 2\pi, r_0)$; since, the differential surface area in cylindrical coordinates can be replaced with $\sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1} r dr d\theta$ [21], the problem can be presented in the new general mathematical form as:

$$\begin{aligned}
&Min_{f_r, f_\theta} \quad I(f) = \int_D f_0(\theta, r, z)\sqrt{f_r^2 + \frac{1}{r^2} f_\theta^2 + 1} \quad rd\theta dr \\
&\quad S.\ to:\ \partial C = S \cup D, D \subset R^2 \quad is \quad given; \\
&\quad S \geq \Psi \quad in\ \Omega_0 \quad (\Psi \quad given, \quad \Omega_0 \subseteq D) \\
&\qquad \int_C rd\theta dr dz = L; \\
&f(\theta_0, r_0) = f(\theta_0 + 2\pi, r_0) = z_0, \quad z_{min} \leq z \leq z_{max}.
\end{aligned} \tag{2}$$

2.1. Transforming into an optimal control problem

We remind, that without loss of generality, by just moving plane $z = 0$ to the plane $z = z_{min}$, we can assume $z_{min} = 0$; then, in order to simplify the calculations, we can change the volume integrals in (1) into surface integrals as follows:

$$\int_C rdzdrd\theta = \int\int_D \int_0^{z=f(r,\theta)} rdzdrd\theta = \int\int_D zrdrd\theta = L.$$

Now, to solve the OS problem (2), we transform the problem into a control one by defining artificial controls $u_i : D \longrightarrow R, i = 1,2,3$ as follows:

$$u_1 = f_\theta, u_2 = f_r, \quad u_3 = f_{r\theta}.$$

We assume that control functions u_1, u_2 and u_3 take their values in U_1, U_2 and U_3, respectively; also suppose that the path function is $z = f(\theta, r) : D \longrightarrow A \subset R$ and we define $\Omega = D \times A \times U_1 \times U_2 \times U_3$, where U_i's are bounded subsets of R.

Definition 2.1. We say the quaternary $P = (z, u_1, u_2, u_3)$ is admissible if:
(1) function $z = f(\theta, r)$ is absolutely continuous and closed (i.e. $f(\theta_0, r_0) = f(\theta_0 + 2\pi, r_0) = z_0$);
(2) control functions u_1, u_2 and u_3 are Lebesgue measurable;
(3) the set of all admissible quaternaries is denoted by W.

Theoretically, the artificial controls are not dependent; this fact should be considered in the solution method, specially when numerical works must be done. For this regard, let $G(\theta, r, z)$ be a continuous function on $D \times A$, we have:

$$\begin{aligned}
&\frac{\partial G}{\partial \theta} = \frac{\partial G}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial G}{\partial z} f_\theta; \\
&\frac{\partial G}{\partial r} = \frac{\partial G}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial G}{\partial z} f_r, \quad \forall G \in C(D \times A).
\end{aligned}$$

Based on what was explained in [10], we can select G functions as polynomials due to the density property of polynomials over vector space $C(D \times A)$ and $A = [0, z_{max}]$; without loss of generality, these functions can be defined as the multiplicative different powers of θ, r, z.

Now for the given D, problem (3) can be rewritten as an optimal control problem:

$$\begin{aligned}
&Min_{P \in W}: \quad I(P) = \int\int_D f_1(\theta, r, z, u_1, u_2, u_3) r d\theta dr \\
&S.\ \ to: \quad \int\int_D zrdrd\theta = L; \\
&S \geq \Psi \quad in\ \Omega_0 \\
&\frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial z} f_\theta = 0, \qquad \forall G \in C\left(D \times A\right); \\
&\frac{\partial G}{\partial r} - \frac{\partial G}{\partial z} f_r = 0, \qquad \forall G \in C\left(D \times A\right); \\
&f\left(\theta_0, r_0\right) = f\left(\theta_0 + 2\pi, r_0\right) = z_0;
\end{aligned} \tag{3}$$

where $f_1(\theta, r, z, u_1, u_2, u_3) \equiv f_0(\theta, r, z)\sqrt{(\frac{1}{r^2}u_1^2 + u_2^2 + 1)}$.

We end this section by reiterating that, as mentioned in [10], the curl of a function $g \in C(R^3)$ is defined as:

$$\nabla \times g = \frac{1}{r}\begin{vmatrix} \hat{r} & r\widehat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ g_r & rg_\theta & g_z \end{vmatrix} = \left(\frac{1}{r}\frac{\partial g_z}{\partial \theta} - \frac{\partial g_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r}\right)\widehat{\theta} + \frac{1}{r}\left(\frac{\partial\left(rg_\theta\right)}{\partial r} - \frac{\partial g_r}{\partial \theta}\right)\hat{z},$$

where $(\hat{r}, \hat{\theta}, \hat{z})$ is the unit vector in cylindrical coordinates. In [6], a novel method is proposed to approximate the solution of nonlinear control problems. The proposed approach (embedding method) is practical and accurate enough; moreover, accuracy could be improved as far as desired. The reader can see [10] for knowing more about the history and applications of this method. Now, in this paper, we will develop this method to solve the obstacle problem (3). The solution method, which is based on an embedding process, involves several stages to set up a linear programming problem whose solution converges to the solution of the original problem. It is one of the outstanding advantages of the method even for strongly nonlinear problems; for more details of the solution procedure, one can see [6].

3. Embedding procedure

Following the mentioned theoretical measure method, to embed problem (3) in a measure space, admissibility of quaternaries $P = (z, u_1, u_2, u_3)$ should be determined in a variational form.

As shown in [10], these properties can be presented by the following variational equalities:

$$\begin{aligned}&\int\int_S \nabla \times F.n\, d\sigma = \int\int_D \nabla \times F.\nabla f\ dA = \int\int_D \Phi^g r dr d\theta \\ &\qquad = r_0(\varphi\left(\theta_0 + 2\pi, r_0, f\left(\theta_0 + 2\pi, r_0\right)\right) - \varphi\left(\theta_0, r_0, f\left(\theta_0, r_0\right)\right)) \equiv d_\Phi; \\ &\nabla \times F = (\tfrac{1}{r}\tfrac{\partial \varphi_{1_z}}{\partial \theta} - \tfrac{\partial \varphi_{1_\theta}}{\partial z})\hat{r} + (\tfrac{\partial \varphi_{1_r}}{\partial z} - \tfrac{\partial \varphi_{1_z}}{\partial r})\hat{\theta} + \tfrac{1}{r}(\tfrac{\partial (r\varphi_{1_\theta})}{\partial r} - \tfrac{\partial \varphi_{1_r}}{\partial \theta})\hat{z}.\end{aligned}$$

where

$$\Phi^g\left(\theta, r, z, u_1, u_2, u_3\right) \equiv \varphi_{z\theta} f_r - \frac{1}{r}\varphi_z f_\theta - \varphi_{rz} f_\theta,$$

B is a sphere so that $D \times A \subset B$, $F = (\varphi_{1_r}, \varphi_{1_\theta}, \varphi_{1_z})$, $\varphi_1 \in D \times A$ and $\Phi \in \Omega$. Function $\varphi(\theta, r, z)$ has continuous partial derivatives with the assumption that controls u_1, u_2 and u_3 are Lebesgue measurable. We remind that variable r is nonzero over the surface, because this variable only takes zero at the origin.

Since the surface equation is $z = f(r, \theta)$, one can conclude that $\nabla f = (-f_r, \frac{-1}{r} f_\theta, 1)$; according to Stoke's theorem, we have:

$$\begin{aligned}&\oint_{\partial D} F dr = \int\int_S \nabla \times F.n\ d\sigma = \int\int_D \nabla \times F.\nabla f\ dA = \int\int_D \Upsilon\left(\theta, r, z, u_1, u_2, u_3\right) r dr d\theta \\ &= \int\int_D \frac{1}{r}\left(2\left(r-1\right) f_r \psi_\theta + \psi f_\theta + f\psi_\theta + \left(r-1\right)\left(f_{r\theta}\psi + f_\theta \psi_r + f\psi_{r\theta}\right)\right) r dr d\theta = 0;\end{aligned}$$

therefore

$$\Upsilon\left(\theta, r, z, u_1, u_2, u_3\right) = \frac{1}{r}\left(2\left(r-1\right) f_r \psi_\theta + \psi f_\theta + f\psi_\theta + \left(r-1\right)\left(f_{r\theta}\psi + f_\theta \psi_r + f\psi_{r\theta}\right)\right).$$

The third class of functions in $C'(B)$ are selected as functions that only depend on the independent variables θ and r; we indicate the set of these functions with $C_1(B)$. In this case, we have:

$$\int_S \frac{1}{\sqrt{\frac{1}{r^2}u_1^2 + u_2^2 + 1}} f\left(\theta, r\right) d\sigma = \int_D f\left(\theta, r\right) r dr d\theta \equiv a_f, \quad f \in C_1\left(B\right)$$

where a_f is the integral of $f(\theta, r)$ on D.

Let $F \in C(\Omega)$ and consider the mapping:

$$\Lambda_p : F \in C(\Omega) \to \int\int_D F(\theta, r, z, u_1, u_2, u_3) r d\theta dr, \tag{4}$$

for any admissible quaternary $p = (z, u_1, u_2, u_3)$.

(1) Λ_p is well defined as the integral of function F which is continuous and finite.

(2) Λ_p is linear, i.e. $\Lambda_p(\alpha F + \beta G) = \alpha\Lambda_p(F) + \beta\Lambda_p(G)$.

(3) Λ_p is positive; i.e. if $F \geq 0$, then $\Lambda_p(F) \geq 0$.

(4) Λ_p is continuous [22]; i.e. $|\Lambda_p(F)| \leq K sup|F(\theta, r, z, u_1, u_2, u_3)|$. Since R^6 is a locally compact space, by the Heine–Borel theorem [22], Ω is a compact Hausdorff space; therefore, for every given p, Riesz Representation Theorem [22] indicates uniquely a positive Radon measure, say $\mu_p \in M^+(\Omega)$, so that:

$$\Lambda_p(F) = \int_\Omega F d\mu_p \equiv \mu_p(F). \tag{5}$$

Therefore, one can embed problem (3) into a measure space by:

$$(z, u_1, u_2, u_3) \in P \mapsto \mu_p \in M^+(\Omega);$$

where $M^+(\Omega)$ is the set of all positive Radon measures on Ω. Regarding the same mentioned difficulties in [6] chapter 2, we shall now enlarge the underlying space and simply consider all measure μ in $M^+(\Omega)$ which satisfying the above mentioned properties (not only those indicated by Riesz Representation Theorem); over this new and larger set called Q. So, one can define the theoretical measure problem (6) instead of the classical problem (3) as follows:

$$\begin{aligned}
&\underset{Q}{Inf} \quad I(P) = \mu\left(f_1\right) \\
&S.\,to: \ \mu(\Phi^g) = d_\Phi, \qquad \varphi \in C(D \times A); \\
&\qquad \mu(\Upsilon) = 0, \qquad \psi \in \mathfrak{I}(D^0); \\
&\qquad \mu(f) = a_f, \qquad f \in C(D); \\
&\qquad \mu(G_1) = 0, \qquad G_1 \in C(D \times A); \\
&\qquad \mu(G_2) = 0, \qquad G_2 \in C(D \times A); \\
&\qquad \mu(\Psi) \le \beta, \\
&\qquad \mu(z) = L,
\end{aligned} \tag{6}$$

where $G_1 = \frac{\partial G}{\partial \theta} - \frac{\partial G}{\partial z} f_\theta$ and $G_2 = \frac{\partial G}{\partial r} - \frac{\partial G}{\partial z} f_r$ and the fifth condition is obtained from the variational form of the obstacle condition (3). Existence solution of (6) is proved by putting weak* topology on $M^+(\Omega)$. It may seem that (6) only gives us a lower bound for the original problem; But apart from the existence and globality of the new solution method, we can come to the following conclusion, for any $\varepsilon > 0$ which is proved in [6]; that is for any $P \in W$ and any ε, there exists a measure $\mu \in Q$, such that:

$$|\Lambda_p(F) - \mu(F)| < c\varepsilon, \qquad \forall F \in C(\Omega).$$

Additionally, there exists an admissible sequence $\{P_j\} = \{(z_j, u_{1j}, u_{2j}, u_{3j})\}$ such that $I(P_j) \to \inf_Q \mu(f_1)$ when $j \to \infty$.

Now, considering the evidence and the method presented in [10], to approximate infinite LP (6) into a finite one. Therefore, now we are looking for an approximation solution of (6) close enough to the optimal solution.

4. Approximation

Consider the minimization the objective function of problem (6) over a subset of Q, called $Q(M_1, M_2, \ldots, M_6)$, which is defined by only a finite number of constraints. This can be achieved by selecting countable sets of functions so that their linear combinations are dense in the appropriate spaces and then by choosing a finite number of such functions. Let sets $\{\varphi_i : i \in N\}, \{\psi_h : h \in N\}, \{f_{sk} : s, k \in N\}, \{G_{1t} : t \in N\}$ and $\{G_{2l} : l \in N\}$ be total sets of functions in the appropriate spaces. We choose a finite number of functions in each of these sets (M_1 number of functions φ_i, M_2 number of functions ψ_h, …and M_6 number of functions G_{2l}); then, in the sense of uniform convergence topology [20], the solution of problem (6) can be approximated by the solution of the problem generated with this action.

By proposition III.1 in [6], if M_1, M_2, M_3, M_4, M_5 and M_6 tend towards infinity, the solution of approximated problem will converge to that of problem (6). Even though the constraints are finite and the problem is a linear semi-infinite programming problem (LSIP), the solution space is not finite. Different methods are available to solve a LSIP problem and its dual problem; but, a positive duality gap may occur between the value of an LSIP problem and its dual [23]. If the set Q is nonempty, the infimum is attained at a measure which is a positive combination of a finite number of measures in $M^+(\Omega)$ which are primary, that is, take only values 0 and 1 [24]. From Proposition III.2 in [6], if f_1 is the Lipschitz function and $\Phi_i^g, \Upsilon_h, f_{sk}, G_{1t}$ and G_{2l} are bounded functions, the optimal measure μ^* in the set $Q(M_1, M_2, M_3, M_4, M_5, M_6)$ has the form $\mu^* = \sum_{j=1}^N \alpha_j^* \delta(q_j^*)$, where $N = M_1 + M_2 + M_3 + M_4 + M_5 + M_6$, with $q_j^* \in \Omega$ and $\alpha_j^* \ge 0$ for $j = 1, 2, \ldots, N$ (here $\delta(q)$ is a unitary atomic measure, characterized by $\delta(q)F = F(q)$ for $F \in C(\Omega)$ (see [20])). Thus, the measure-theoretical optimization problem is equivalent to a nonlinear optimization problem in which the unknowns are coefficients α_j^* and supports $\{q_j^*\}$. It would be much more convenient if we could minimize the function $\mu \mapsto \mu(f_1)$ only with respect to the coefficients α_j^*, which would result in a finite linear programming problem. However, we do not know the supports of the optimal measure. The answer lies in approximation of this support by introducing a dense subset in Ω; let D'_Ω be a countable dense subset of Ω; then, as a consequence of Proposition III.3 in [6], there exist a measure $\mu \in M^+(\Omega)$ with the form $\sum_{j=1}^M \alpha_j \delta(q_j)$ such that $q_j \in D'_\Omega$. This leads us to discretize Ω by nodes $q_i, i = 1, 2, \ldots, M$ lying in a countable dense subset of it and setup the following finite linear programming:

$$\begin{aligned}
&Min \qquad \mu(f_1) = \textstyle\sum_{j=1}^M \alpha_j f_1\left(q_j\right) \\
&S.\,to: \textstyle\sum_{j=1}^M \alpha_j \Phi_i^g\left(q_j\right) = d_{\Phi_i}, \qquad i = 1, 2, \ldots, M_1; \\
&\textstyle\sum_{j=1}^M \alpha_j \Upsilon_h\left(q_j\right) = 0, \qquad h = 1, 2, \ldots, M_2; \\
&\textstyle\sum_{j=1}^M \alpha_j f_{sk}\left(q_j\right) = a_{f_{sk}}, \quad s = 1, 2, \ldots, M_3, \ k = 1, 2, \ldots, M_4; \\
&\textstyle\sum_{j=1}^M \alpha_j G_{1t}\left(q_j\right) = 0, \qquad t = 1, 2, \ldots, M_5; \\
&\textstyle\sum_{j=1}^M \alpha_j G_{2l}\left(q_j\right) = 0, \qquad l = 1, 2, \ldots, M_6; \\
&\textstyle\sum_{j=1}^M \alpha_j \Psi(q_j) \le \beta, \\
&\textstyle\sum_{j=1}^M \alpha_j z_j = L\,.
\end{aligned} \tag{7}$$

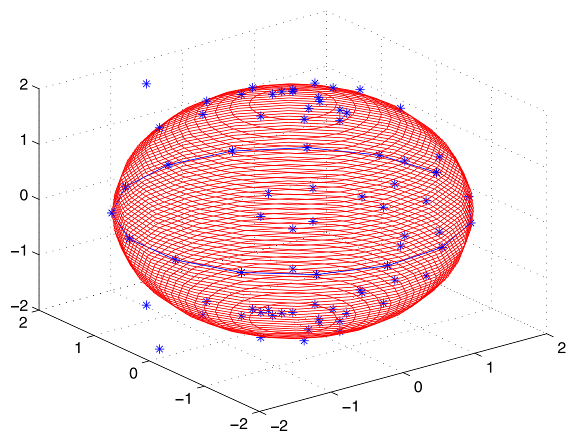

Fig. 2. Optimal surface for Example 1 without rejecting the outliers.

In this manner, the optimal solution to (7) tends to the optimal solution of problem (6) where $M_1, M_2, \ldots, M_6$ and M tend to infinity [25] (in paper [10], the convergence of this method was proved by 3 theorems and the solution algorithm will be presented in sub-section 4.2). One may say that, at the moment, we are faced with a large-scale problem which may have its own difficulties. We emphasize that there are two other points worthy of notice. Firstly, given the particular choice of f_{sk} functions and the right-hand-side value of the second class of constraints, most of the elements of the coefficient matrix are zero. This, then, reduces the number of computations and makes the coefficient matrix sparse. Secondly, methods such as the interior point in solving linear programming problems for sparse matrices simplify the process of solving the problem by limiting the number of iterations and by saving time [26].

According to [10], we shall explain how one can choose the total set of functions for the constraints in (6). Let set φ_i be such that the linear combinations of these functions are uniformly dense (i.e. they are dense in the topology of uniform convergence) in space $C'(B)$; these functions can be taken to be monomials in the components of variables r, θ, z. For the second set of equations in (6), ψ_j's are from the following classes of functions with compact support:

$$\psi_1 = (r - h(\theta))(sin(i\pi\theta)), \qquad i = 1, 2, 3, \ldots;$$
$$\psi_2 = (r - h(\theta))(cos(i\pi\theta)), \qquad i = 1, 2, 3, \ldots;$$
$$\psi_3 = (r - h(\theta))(cos(i\pi\theta)sin(i\pi\theta)), \qquad i = 1, 2, 3, \ldots;$$

these functions are useful as bases for Fourier series, which are usually utilized in analyzing physics and engineering problems. The linear combinations of this set of functions are dense in the approximated spaces and we choose only M_2 number of them. Also, we consider the third set of Eq. (7) as follows:

$$f_{sk}(\theta, r) = \begin{cases} 1, & if \quad \theta \in J_{1s},\ r \in J_{2k}; \\ 0, & otherwise, \end{cases}$$

where $J_{1s} = [\frac{(2\pi)(s-1)}{M_3}, \frac{(2\pi)(s)}{M_3})$ and $J_{2k} = [\frac{(h(\theta))(k-1)}{M_4}, \frac{(h(\theta))(k)}{M_4})$; then

$$\int\int_D f_{sk}(\theta, r)\, d\theta dr = \int_{r_{k-1}}^{r_k}\int_{\theta_{s-1}}^{\theta_s} r d\theta dr = \frac{1}{2}\left(\theta_s - \theta_{s-1}\right)\left({r_k}^2 - {r_{k-1}}^2\right) \equiv a_{f_{sk}}.$$

Although these functions are not continuous, the linear combinations of them can efficiently and properly approximate any functions in $C(\Omega)$ (see [22]).

5. A note on convergence

In this section, we investigate the convergence of the new proposed method according to the following 3 propositions. It is mentioning that in order for problem (1) to be convergent, the functions involved should be continuous and control variables are bounded.

Proposition 5.1. *The transformation $P \to \Lambda_p$ of an admissible quaternary in W into the linear mapping Λ_p defined in (4) is an injection.*

Proof. We must show that if $P_1 \neq P_2$ then, $\Lambda_{p_1} \neq \Lambda_{p_2}$. Indeed, if $P_1 = (z_1, u_{11}, u_{21}, u_{31})$ and $P_2 = (z_2, u_{12}, u_{22}, u_{32})$ are different admissible quaternary, a continuous positive function F can be constructed on $C(\Omega)$ so that, the right-hand side of Λ_{p_i} corresponding to P_1 and P_2 are not equal. Then, the linear functionals are not equal.

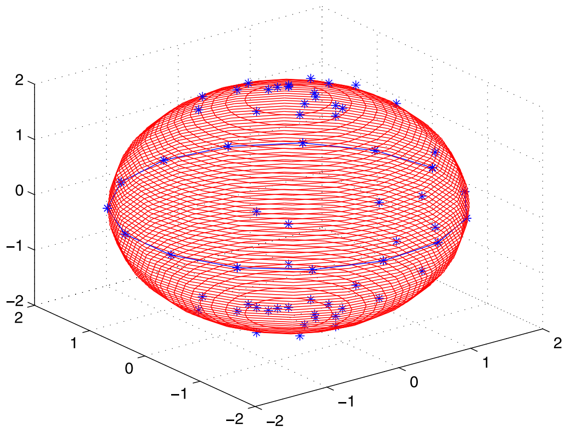

Fig. 3. optimal surface for Example 1 with rejecting the outliers.

Proposition 5.2. *Let $Q(M_1, M_2, \dots, M_6)$ be a subset of $M^+(\Omega)$ consisting of all measures which satisfy constraints (7). As $M_1, M_2, \dots, M_6$ tend to infinity, then,*

$$\underset{Q(M_1,M_2,\dots,M_6)}{Inf} \quad \mu(f_1) \longrightarrow \underset{Q}{Inf} \quad \mu(f_1).$$

Proof. Regarding the density properties of the selected subspaces of appropriated spaces $C'(B)$, the proof is similar to the proof of Proposition III.1 in [6].

Proposition 5.3. *Measure μ^* in set $Q(M_1, M_2, \dots, M_6)$ at which the function $\mu \to \mu(f_1)$ attains its minimum has the following form*

$$\mu^* = \sum_{j=1}^{M} \alpha_j^* \delta(q_j^*)$$

where δ is an atomic measure, $q_j^* \in \Omega$ and $\alpha_j^* \geq 0, \quad j = 1, 2, \dots, N$.

Proof. The above-mentioned proof is similar to that of Proposition III.2 in [6].

Now, we prove that problem (7) is equivalent to problem (1) when $M, M_1, M_2, \dots, M_6 \to \infty$.

We remind that, problem (2) is the same as problem (1) which was presented in cylindrical coordinates. Then, extra constraints were added and problem (3) was resulted; these constraints are necessary for a better communication between control variables and they also show the admissibility of the quaternaries. On the other hand, according to Proposition 5.1, problem (3) equals problem (2). Furthermore, since set W of admissible quaternary can be considered (by means of the injection transformation in Proposition 5.1) as a subset of Q, the minimization of problem (6) is global; that is, the global infimum of problem (6) can be approximated well [6]. Also, problem (6) has at least one solution. Now, according to Propositions 5.2 and 5.3, when $M_1, M_2, \dots, M_6 \longrightarrow \infty$, problem (7) is equivalent to (6). Also, with respect to Proposition III.3 and Theorem III.1 in [6], problem (7) has a solution and this solution converges to the solution of problem (6) when M is sufficiently large. So, problem (1) has at least one solution and its solution can be determined with the method presented in the next section.

6. Algorithm (solution procedure)

To apply the mentioned method for obstacle problems to achieve the optimal surface, some other actions must be performed as well. Here, we present an algorithmic path for determining the solution procedure.

Step (1) Statement of the classical problem to integral equations: Using relationship (4), all conditions as well as the objective function of (3) and the obstacle condition are expressed in the form of integral relationships over region D.

Step (2) Transferring the problem to control problem and measures: By defining artificial controls based on the obligatory function presentation of the unknown optimal surfaces, first the problem is transferred to an optimal control one. Then, according to the Riesz Representation Theorem, by a one-to-one correspondence, the problem is embedded in a subset of positive Radon measures. Then, the underlying space is enlarged (as explained in Section 3) and we setup problem (6).

Step (3) Identification of the optimal solution: By considering a finite number of constraints in (6), we change the new problem into an approximated finite LP problem by applying the mentioned total sets and discretization schemes. Indeed, the given sets

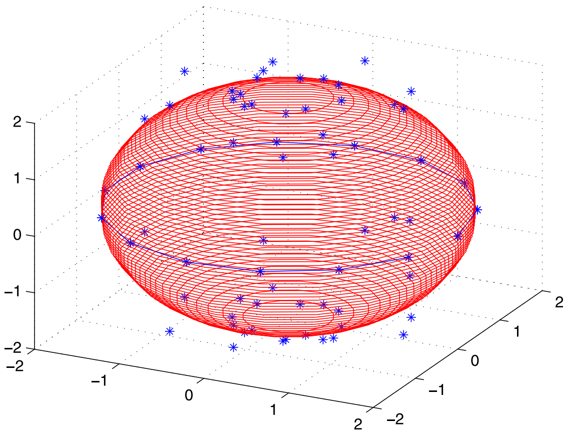

Fig. 4. Optimal surface for Example 2 without rejecting outliers.

D, A, U_1, U_2 and U_3 are divided into $n_1, n_2, \ldots, n_6$ equal parts respectively, so that $N = n_1.n_2.n_3.n_4.n_5.n_6$ number of 6-dimensional cells in Ω is obtained. Then, arbitrary points $q_i = (\theta_i, r_i, z_i, f_{\theta_i}, f_{r_i}, f_{r\theta_i})$ are selected in each of these 6-dimensional cells, respectively. Now, one is able to set up the finite LP (7) with N variables and $M = M_1 + M_2 + (M_3 \times M_4) + M_5 + M_6 + 1$ constraints. In this regard, by discretization on D and Ω_0, the obstacle condition $S \geq \psi$ is also considered; this condition is applied if it is necessary.[1]

Step (4) Identifying the optimal shape: From the solution of the finite LP in the previous step (problem (7)), we identify the indices n such that the components α_n^* of the extreme points are positive; then, the corresponding values θ_n and r_n associated with them (in J_{1s} and J_{2k}) are introduced, as mentioned in [6]. We then partition this subinterval into further subintervals, one for each index n with these properties, of the length equal to the component α_n^*, and make $\theta = \theta_n$, $r = r_n$, $u_1(r,\theta) = u_{1n}$, $u_2(r,\theta) = u_{2n}$, $u_3(r,\theta) = u_{3n}$ and $z(r,\theta) = z_n$ in these subintervals. These subintervals which partition J_{1s} and J_{2k} can be put together in any order [25]. In this regard, some points of the optimal surface are determined.

Step (5) Drawing the optimal surface: To represent the nearly optimal surface, by using the curve fitting (for instance from toolbox of MATLAB software), we fit a surface to these points in Cartesian coordinates. In this step, due to the employment of the approximation schemes, some outlier points may occur between the obtained optimal points from Step 4, which may make some tribulation in the smoothness. Therefore, by using a suitable outlier detection algorithm (below), one can reject the outliers before the curve fitting, to achieve a more desirable shape.

6.1. Outlier detection among nearly optimal points

Suppose we have a data set such that except for a number of its members, all of them belong to special groups, or clusters. Those members which do not belong to any cluster are called outliers or in a very simple terms, an outlier refer to data which have a meaningful distance from the others (majority data in the data set). Thus, the data set can be divided into two categories: consistent and outlier data [27]. Outliers are extreme values that lie near the limits of the data range or go against the trend of the remaining data. Identifying outliers is important because they may represent errors in data entry and also may deliver unstable results [28,29].

A variety of methods are available to detect outliers. But, in general, all methods could be classified into two categories: labeling methods that each data is assigned one of the two labels *consistent* or *outlier*, scoring method that assigns a number (called the inconsistency factor) to all data. The second kind is more flexible because one can choose a threshold value for the incompatibility factors. In numerical examples, in order to reduce the approximation error, we used LoOP (Local Outlier Probability) algorithm [29] to reject the outliers. Here, we reject the data whose corresponding outlier factors are more than 0.4.

Local outlier probability algorithm: Assume $o \in A$ is an arbitrary member of data set A and $V \subseteq A$ includes element o that is called context set. For every natural number k, context set V contains o and the k number closest points to o in other words, if we assume $dist_k(o)$ as distance between o and its kth nearest neighbor; we consider V as follow:

$$V = \left\{ s \in A : \quad d(s,o) \leq dist_k(o) \right\};$$

Definition 6.1. For a given set V, the standard deviation of o from $s \in V$ is defined as follow:

$$\sigma(V,o) = \sqrt{\frac{\sum_{s\in V} d^2(s,o)}{|V|}};$$

[1] As mentioned in Section 2, some obstacle conditions could be related to the discretization schemes; for instance if the obstacle is $z \geq \beta$ in Ω_0, then in discretization, for choosing points (θ, r, z) when $(\theta, r) \in \Omega_0$, z is chosen in $[\beta, z_{max}]$.

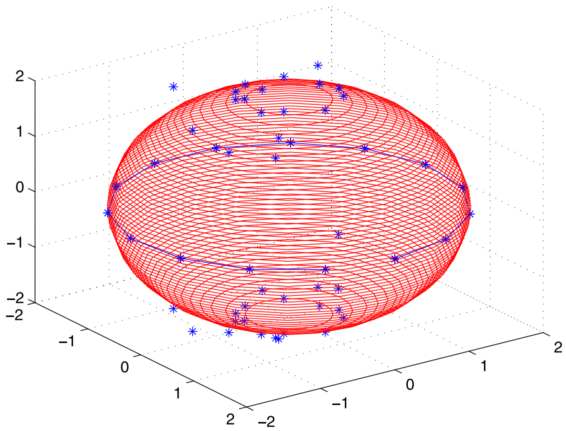

Fig. 5. Optimal surface for Example 2 with rejecting outliers.

Definition 6.2. For a constant parameter $\lambda = 1, 2$ or 3, we define:

$$Pdist\,(\lambda, V, o) = \lambda \times \sigma\,(V, o)$$

Definition 6.3. Density of the set V with respect to o is defined as follow:

$$PLOF\ \ (\lambda,\ V, o) = \frac{Pdist\,(\lambda, V, o)}{E_{s\in V}(Pdist\,(\lambda, V, s))} - 1$$

where the denominator means expected (arithmetic mean or center of gravity) is calculated as:

$$E_{s\in V}\,(Pdist\,(\lambda, V, s)) = \frac{1}{|V|}\sum_{s\in V} Pdist\,(\lambda, V, s)\,;$$

Definition 6.4. A standard deviation for different amounts of $PLOF\ \ (\lambda,\ V, o)$ is defined as follow:

$$nPLOF = \lambda.\sqrt{E[(PLOF(\lambda, V, o))^2]} = \frac{\lambda}{\sqrt{|\ V|}}\left(\sum_{s\in V}(PLOF(\lambda, V, o))^2\right)^{\frac{1}{2}}$$

Definition 6.5. Probabilistic factor or LoOP local inconsistency is defined as:

$$LoOP\,(o) = \max\left\{\ 0, \mathrm{erf}\left(\frac{PLOF\,(\lambda, V, o)}{nPLOF.\sqrt{2}}\right)\ \right\}$$

where:

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

The flowchart of LoOP algorithm used in this paper is shown in Fig. 9.

7. Numerical examples

First, it is worth mentioning that in all present examples, the related LPP has been solved using MATLAB 9.8 software, Revised Simplex Method, and the figures have been obtained using MATLAB toolbox for fitting the surface.

Generally, a presented method for solving an obstacle problem should be able to solve the OS problems as well, since if the obstacle condition in (3) is removed, we face with an OS problem. In this regard, in this section, we present four examples; the first two are in the absence of an obstacle, and the next two are in the presence of one. Examples 1 and 2 are well-known examples in geometry and calculus of variations which are given to examine the efficiency of the new method in 3-D. Example 3 is a general engineering problem which has been solved by projection [30], active set strategy [31], multigrid and multilevel [32], level set [2], piecewise linear system [33], discontinuous Galerkin [34], moving obstacle to approach the contact and the meshless methods [3]. The last one is minimization of the surface area with presence of obstacle.

Example 1. It is a very well-known fact that of all 3-D objects with the same volume, the sphere has the least surface area. To achieve this fact with the new method, we want to minimize the area of an unknown shape with a specified physical volume.

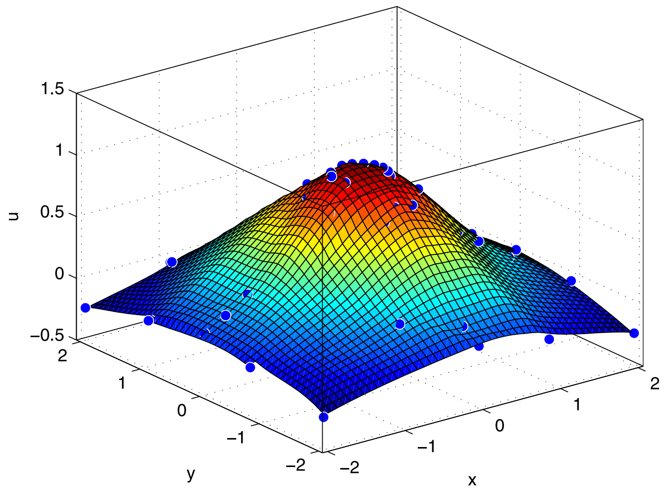

Fig. 6. Optimal surface for Example 3 with obstacle $\sqrt{1-x^2-y2}$.

Supposing that the optimal shape is symmetrical with respect to (r,θ)- plane and we specify the optimal points just for $z>0$; then, we can later add the symmetrical points at the other side of the plane to obtain the whole optimal shape. We know that the real optimal shape is a sphere with radius 2 and volume $16\pi/3$. Thus, the shadow of the unknown surface is a circle with radius 2 and the goal is to minimize $I=\int\int_S d\sigma=\int\int_D\sqrt{f_r^2+\frac{1}{r^2}f_\theta^2+1}\ \ rdrd\theta$, subject to $\int\int\int_C dV=\frac{16\pi}{3}$. Provided that there is also a given fixed point on the boundary of region D as $(2\pi,2,0.01)=(0,2,0.01)$ and $z_{max}=2,\ \ z_{min}=0,\ 0\le\theta\le 2\pi,\ 0\le r\le 2$; by trial and error we choose $-1/5\le f_\theta\le 1/5,\ 0\le f_r\le 1,\ -1\le f_{r\theta}\le 1$.

To discretize $\Omega=D\times A\times U_1\times U_2\times U_3$, we choose $M=16\times 6\times 10\times 8^3$ points in this set as follow; selecting 8 points in U_1 for f_θ as: $\frac{-3}{2},\frac{-15}{14},\frac{-9}{14},\frac{-3}{14},\frac{3}{14},\frac{9}{14},\frac{15}{14},\frac{3}{2}$; 8 points in U_2 for f_r as: $0,\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7},1$; 8 points in U_3 for $f_{r\theta}$ as: $-1,\frac{-5}{7},\frac{-3}{7},\frac{-1}{7},\frac{1}{7},\frac{3}{7},\frac{5}{7},1$; 16 angles in $[0,2\pi]$ for θ as: $0,\frac{2\pi}{15},\frac{4\pi}{15},\frac{6\pi}{15},\frac{8\pi}{15},\frac{10\pi}{15},\frac{12\pi}{15},\frac{14\pi}{15},\frac{16\pi}{15},\frac{18\pi}{15},\frac{20\pi}{15},\frac{22\pi}{15},\frac{24\pi}{15},\frac{26\pi}{15},\frac{28\pi}{15},2\pi$; 10 values in $A=[0,2]$ for z as: $\frac{3}{10},\frac{44}{90},\frac{61}{90},\frac{78}{90},\frac{95}{90},\frac{112}{90},\frac{129}{90},\frac{146}{90},\frac{163}{90},2$; 6 values for r as: $0.3,0.6,0.9,1.2,1.6,2$. Then, M number of nodes $(\theta,r,z,f_\theta,f_r,f_{r\theta})$ in Ω are introduced (we supposed $\pi=3.14159265$ and hence all nodes belong to the dense subset of Ω).

To set up the linear programming problem (7), for the first set of constraints, with $M_1=5$, we choose: $\varphi_1=\theta^2r^3z,\ \varphi_2=\theta^3r^5z,\ \varphi_3=\theta^3r^2z,\ \varphi_4=\theta^2r^2z^2,\ \varphi_5=\theta^2z^2$; for the second set, $M_2=9$, and :

$$\psi_1=(r_n-2)(\sin(n\pi\theta_n)),\ \ \psi_2=(r_n-2)(\cos(n\pi\theta_n));$$
$$\psi_3=(r_n-2)(\cos(n\pi\theta_n))(\sin(n\pi\theta_n)),\qquad n=1,2,3;$$

for the third set, with $M_3=8$ and $M_4=3$ and for the final set $M_5+M_6=2$ we select:

$$G_{1l}=(2\theta_nz_nr_n)-(\theta_n^2r_nf_{\theta_n}),G_{2k}=(2\theta_nz_n^2r_n^2)-(2\theta_n^2z_nr_n^2f_{\theta_n}).$$

Thus, the related linear programming problem (7) (having 41 constraints and $M=16\times 6\times 10\times 8^3$ variables) is solved by using MATLAB 9.8 software and the optimal points (θ_n^*,r_n^*,z_n^*) corresponding to the optimal coefficients $\alpha_n^*>0$ are obtained. Next, we specify the optimal points (x_n^*,y_n^*,z_n^*) by means of the relationships $x_n^*=r_n^*cos(\theta_n^*)$ and $y_n^*=r_n^*sin(\theta_n^*)$. For the first case, we fit a surface to these points without rejecting outliers by using the related MATLAB toolbox. For the second case, we reject the outliers by using the LoOP algorithm and obtaining the optimal surface by doing the same curve fitting (Figs. 2 and 3). Furthermore, to calculate the average error between the analytical solution and the obtained result, we use $Error=\sqrt{\frac{1}{n}\sum_{i=1}^n(\rho_{real}-\rho_{numerical})_i^2}$ (see [2]); where $\rho_{real}=2$ (radius of the real sphere) and $\rho_{numerical}=\sqrt{x_i^2+y_i^2+z_i^2}$ (radius of the obtained sphere passing points (x_i,y_i,z_i)). The error was equal to 0.1898 for the case with existing outliers and 0.0644 for the case without them. Further, correlation between the component z_n obtained from the solution of the related linear programming (7) and the same component of the real sphere were computed. These value were 0.9052 and 0.9661 for the first and the second case, respectively. Moreover, the obtained optimal area was 23.0892 which was also close to the real area of the hemisphere, i.e. 8π. Considering the amount of average errors as well as the obtained optimal value of the objective function, the obtained results without outliers were very close to the real solution, and the amount of error was remarkably smaller.

Observing Figs. 2 and 3, one can conclude that in spite of considering only a finite number of constraints, the presented 3-D (obstacle) shape-measure method successfully solves this example. Also, we see that this new method has small errors and also a high correlation between values after rejecting the outliers. Therefore, the obtained results are perfectly satisfactory. Meanwhile, this example showed the ability of the method in obtaining symmetrical optimal shapes as well.

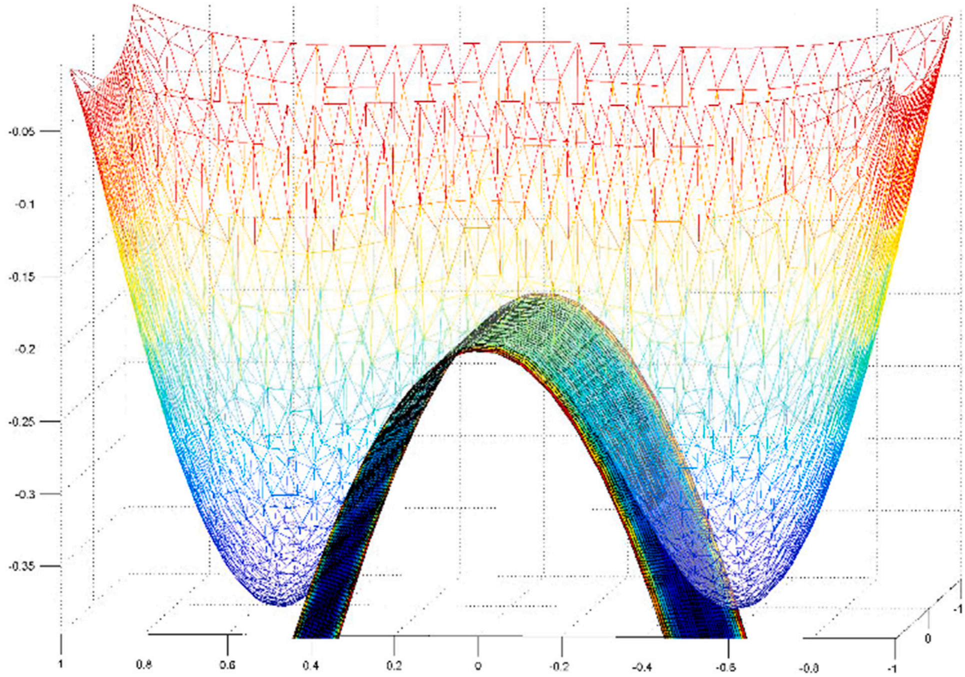

Fig. 7. Optimal surface for Example 3 with obstacle $-x^2$.

Example 2. According to an old problem, among all shapes having the same area, the sphere has the biggest volume. Therefore, in this example, we intend to maximize the volume of an unknown shape on the condition that its area is 8π, which is the area of a hemisphere having a radius of 2. This problem is solved in the same way as explained in Example 1, except that the objective function was $I = \int\int\int_C dV$ with the area constraint $\int\int_D \sqrt{\frac{1}{r^2} f_\theta^2 + f_r^2 + 1}\ \ r dr d\theta = 8\pi$. We also calculated the average error which was 0.2487 without rejecting the outliers and 0.1197 with rejecting them; the correlation between the real z of the sphere and its numerically approximation, z_n, were 0.7043 and 0.9301, respectively. Besides, the obtained nearly optimal objective value was equal to 16 which is close enough to the real value $16\pi/3$. Figs. 4 and 5 present the surfaces fitted based on the obtained optimal points.

Example 3. Now, we intend to solve the obstacle problem; this problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above (or below) a given obstacle [35].

Indeed, we employ the new method for solving the following obstacle problem which has been solved in [2,30–33] through various methods. To this aim, we consider a region in domain D such that the solution to the problem for the points in this region is a constant, while it has values greater than this constant in the rest of the points:

$$\begin{aligned} &Min \quad E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dxdy \\ &S.\ to: \quad v \in k, \qquad k = \{v \in H_0^1(\Omega),\ \ v|_{\partial\Omega} = 0,\ \ v \geq \psi \ \ a.e.\ \ in \ \ \Omega\}. \end{aligned} \tag{8}$$

Because this example is given from [2], and to compare the results, we choose $\Omega = [-2, 2] \times [-2, 2]$ and

$$\psi = \begin{cases} \sqrt{1 - x^2 - y^2}, & for \quad x^2 + y^2 \leq 1; \\ -1, & elsewhere, \end{cases}$$

with the consistent Dirichlet boundary condition (see [2]). The obstacle is a subset of Ω which is a circle with a radius of 1. In addition, for the points in this circle, condition $v \geq \psi$ should hold, whereas at the remaining points in Ω, v can have any value greater than -1. The analytical value of v is equal to:

$$v^*(x, y) = \begin{cases} \sqrt{1 - x^2 - y^2}, & r \leq r^*; \\ \dfrac{-(r^*)^2 Ln\left(\frac{r}{R}\right)}{\sqrt{1-(r^*)^2}}, & r \geq r^*, \end{cases}$$

where $r = \sqrt{x^2 + y^2}$, $R = 2$ and $r^* = 0.69799651482$, which satisfy $(r^*)^2(1 - Ln\left(\frac{r^*}{R}\right)) = 1$ (see [2] for more details). The aim of this problem is to find function $v(x, y)$ so that $E(v)$ is minimized. Therefore, by using the introduced method, we consider $z = v(x, y)$,

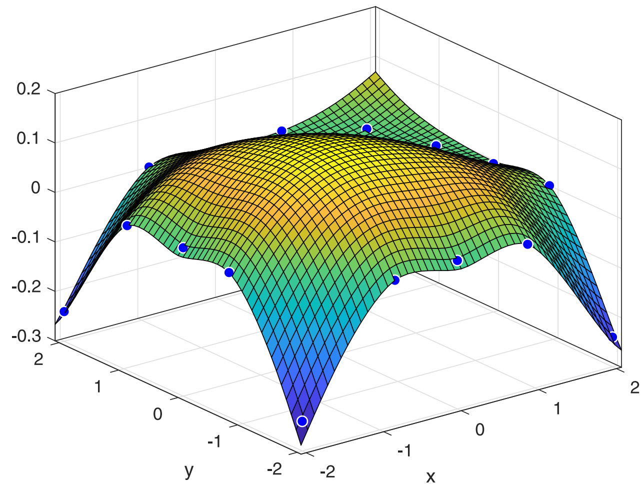

Fig. 8. Optimal surface for Example 4 with obstacle.

rewritten the objective function in cylindrical coordinates as $E(v) = \frac{1}{2}\int_{\Omega}(v_r^2 + \frac{1}{r^2}v_\theta^2 + 1)rdrd\theta$ with Ω as the image of the unknown surface.

To solve this problem, discretization schemes for variables (θ, r, z) and controls (v_r, v_θ) have been done like that of previous examples and the same functions and software are used as well. We compare our obtained results with the results in [2] where the same problem was solved by using the level set method. In that paper, after 7124 iterations, the value of the objective function turned out to be 3.476744 and the average error between the numerical and analytical values was 0.0273. But in our method, the obtained this average error equals to 0.0164 and the obtained objective function value is 2.6042. It is especially significant that we solved this problem without considering any initial shape or values and by just completing one iteration (See Fig. 6). Moreover, the correlation between numerical and analytical values is 0.9828, which is significant. Also, to have another try for the other type of problem (8), we solve this problem using obstacle constraint in the form of $\Psi = -x^2$ again. In the same way as mentioned, the new method is successfully applied and the optimal surface, which is located above the obstacle, is displayed in Fig. 7.

Example 4. Here, we express the obstacle problem in such a way that the objective function is the minimization of the surface area. The aim is to determine a surface whose area is the least possible which is located on the top of obstacle ψ and also, the amount of the energy in this area is equal to a constant number β. Therefore, we have:

$$\begin{array}{ll} Min\ : Z = \int_S d\sigma & \\ S.\,to: & \frac{1}{2}\int_{\Omega}|\nabla v|^2 dxdy = \beta; \\ & v \geq \psi \qquad\qquad in\ \Omega \end{array}$$

Function ψ has been considered like Example 3. Moreover, for this example, we choose $\beta = 2.6042$, the optimal value of the energy function in the first constraint is the optimal amount obtained in the previous example.[2]

This problem is used in designing solar cells. The optimal surface obtained is given (which has been determined based on discretization and the method mentioned in Examples 1 and 3) in Fig. 8. As expected, by considering the definition of the objective function, nearly all of the obtained optimal points located above the obstacle ψ.

8. Conclusion

This paper proposed a novel and practical approach for obtaining the solution to general 3-D obstacle problems. First, the problem was transferred into an optimal control frame in a variational representation. Then, in an algorithmic path, the nearly optimal shape was constructed by transferring the problem into a measure space, extending the underlying space, applying two approximation steps and obtaining the optimal surface from the solution of an appropriate finite linear programming problem. Moreover, the optimal value for the general form of the objective function and the nearly optimal shape were determined by implementing Simplex algorithm perfectly well. Additionally, in this method, a smoother shape was obtained by rejecting the outlier data and smooth fitting procedures. Compared with other methods, this approach is more practicable since the results are obtained by solving one LP. Furthermore, this approach avoids complications such as shape derivative, rate of shape changes, mesh design and initial shape, thereby reducing error. Then, this method gives us a smoother shape by rejecting the outlier data. It is also especially practical and accurate enough for systems with nonlinear terms. Also, accuracy can be improved to whatever extent desired.

[2] We remind that the similar problem is considered.

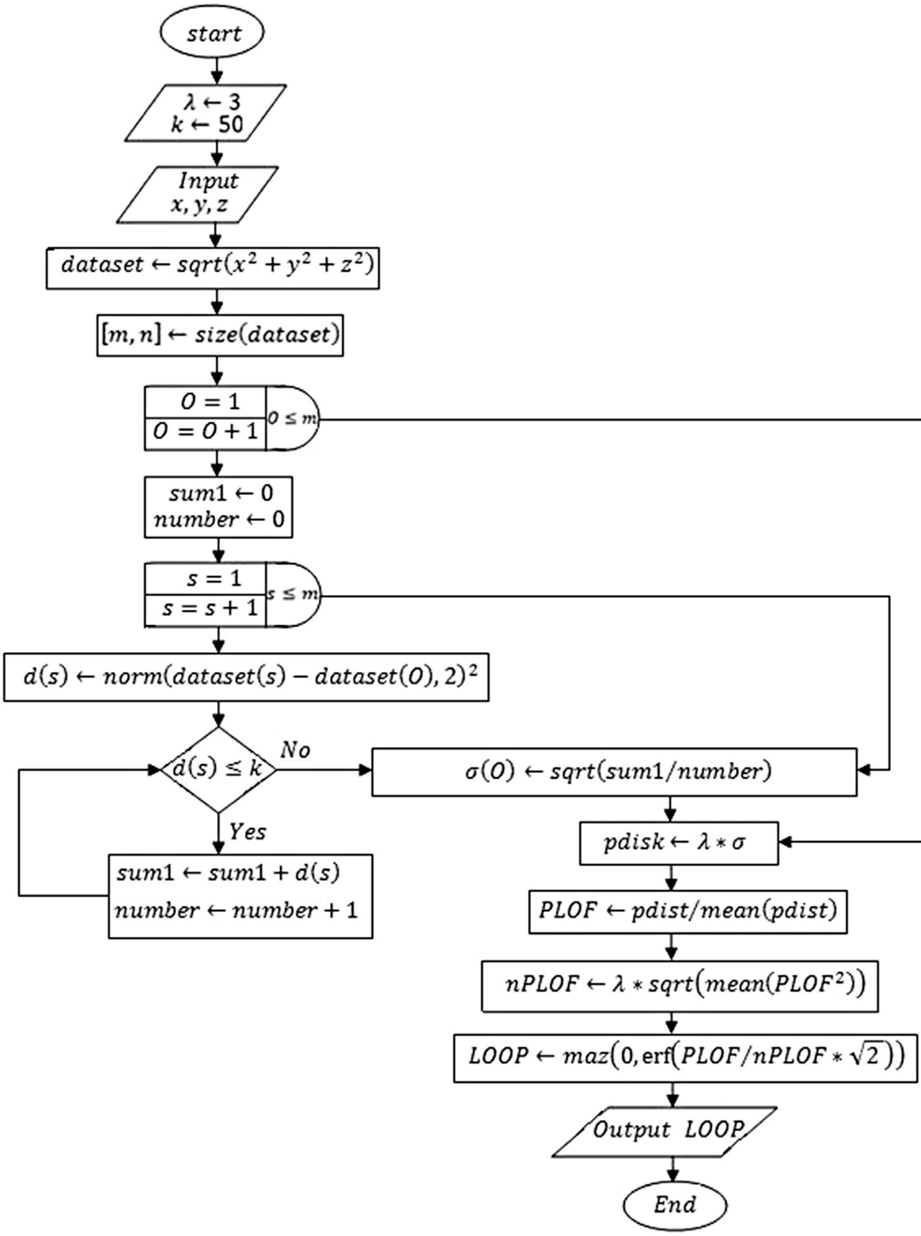

Fig. 9. LoOP flowchart for identifying outlier data.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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